Analysis in Theory and Applications
Anal. Theory Appl., Vol. **31**, No. 2 (2015), pp. 101-108
DOI: 10.4208/ata.2015.v31.n2.1

(L^p,L^q)-Boundedness of Hausdorff Operators with Power Weight on Euclidean Spaces

Guilian Gao[1,*] and Amjad Hussain[2]

[1] *School of Science, Hangzhou Dianzi University, Hangzhou 310018, China*
[2] *Department of Mathematics, COMSATS Institute of Information Technology, Wah Cantt, Pakistan*

Received 3 March 2014; Accepted (in revised version) 24 February 2015

Abstract. In this paper, we prove the (L^p,L^q)-boundedness of (fractional) Hausdorff operators with power weight on Euclidean spaces. As special cases, we can obtain some well known results about Hardy operators.

Key Words: Hausdorff operator, Hardy operator, Cesàro operator, Young's inequality.

AMS Subject Classifications: 42B99, 46E30, 47B38

1 Introduction

Hausdorff operators (Hausdorff summability methods) play important roles in the study of one dimensional Fourier analysis, particularly the summability of classical Fourier series. Modern theory of Hausdorff operators started with the work of Siskakis [30] in complex analysis setting and with the work of Georgakis [16] and Liflyand and Móricz [25] in the Fourier transform setting. One can see [24] for a brief overview of Hausdorff operators. Some recent developments for Hausdorff operators can be founded in [1–8, 12–15, 20–27, 29, 30, 33]. Now we recall the one-dimensional Hausdorff operator

$$h_\Phi f(x)=\int_0^\infty \frac{\Phi(t)}{t} f\Big(\frac{x}{t}\Big)dt,$$

where Φ is a locally integrable function on $(0,\infty)$. Liflyand and Móricz [25] proved that h_Φ generated by a function $\Phi\in L^1(\mathbb{R})$ is a bounded linear operator on the real Hardy space $H^1(\mathbb{R})$. Following this, the boundedness of h_Φ was considered in various spaces, for example, see [3, 20, 26].

*Corresponding author. *Email addresses:* `gaoguilian305@163.com` (G. L. Gao), `ahabbasi123@yahoo.com` (A. Hussain)

 ©2015 Global-Science Press

The one-dimensional Hausdorff operator contains the classical Hardy operator h and its adjoint operator h^* if we choose $\Phi(t)$ as $t^{-1}\chi_{(1,\infty)}(t)$ and $\chi_{(0,1]}(t)$ respectively, i.e.,

$$hf(x):=\frac{1}{x}\int_0^x f(t)dt \quad \text{and} \quad h^*f(x):=\int_x^\infty \frac{f(t)}{t}dt.$$

It is well known that Hardy operators are important operators in Harmonic analysis, for instance, see [18, 19]. On the other hand, if we choose $\Phi(t)=\alpha(1-t)^{\alpha-1}\chi(0,1)(t)$ for $\alpha=1,2,\cdots$, then $H_\Phi=C_\alpha$ is called the Cesàro operator of order α. A brief history of the study of the Cesàro operator can be found in [20].

For multidimensional Hausdorff operators, there are many kinds of definitions [1, 3, 4, 16–18, 21, 22]. One of the interesting definitions of the Hausdorff operators is

$$H_{\Omega,\Phi}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(y')}{|y|^n}\Phi\Big(\frac{x}{|y|}\Big)f(y)dy,$$

where $\Omega(y')$ is an integrable function defined on the unit sphere S^{n-1}. Similar to h_Φ, $H_{\Omega,\Phi}$ contains the high dimensional Hardy operator H and its adjoint operator H^* (see the below definitions).

Recently, Chen, Fan and Li [5] obtained that if Φ is a radial function and $1\le p\le\infty$, then

$$\|H_{\Omega,\Phi}f\|_{L^p(\mathbb{R}^n)}\le\|\Omega\|_{L^{p'}(S^{n-1})}|S^{n-1}|^{\frac{1}{p}}\int_0^\infty\frac{|\Phi(t)|}{t}t^{\frac{n}{p}}dt\cdot\|f\|_{L^p(\mathbb{R}^n)}. \tag{1.1}$$

Particularly,

$$\|H_\Phi f\|_{L^p(\mathbb{R}^n)}\le|S^{n-1}|\int_0^\infty\frac{|\Phi(t)|}{t}t^{\frac{n}{p}}dt\cdot\|f\|_{L^p(\mathbb{R}^n)}.$$

As applications, they obtained the known results about boundedness of Hardy operators H and H^* on $L^p(\mathbb{R}^n)$ (see [9, 10]). For a general function Φ, Wang [31] proved

$$\|H_\Phi f\|_{L^p(\mathbb{R}^n)}\le|S^{n-1}|^{\frac{1}{p'}}\int_0^\infty\Big(\int_{S^{n-1}}\big|\Phi(t\varphi)\big|^p d\varphi\Big)^{\frac{1}{p}}t^{-1+\frac{n}{p}}dt\cdot\|f\|_{L^p(\mathbb{R}^n)}. \tag{1.2}$$

In [29], Lin and Sun defined the n-dimensional fractional Hausdorff operator for a radial function Φ as follows

$$H_{\Phi,\beta}f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|y|^{n-\beta}}\Phi\Big(\frac{|x|}{|y|}\Big)dy,\quad 0\le\beta<n.$$

They gave the sufficient condition on Φ about the boundedness of $H_{\Phi,\beta}$ on $L^p(|x|^\gamma)$, where $0<\beta<n$. If we choose Φ as $|t|^{\beta-n}\chi_{(1,\infty)}(|t|)$ and $\chi_{(0,1]}(|t|)$, $H_{\Phi,\beta}f$ becomes the fractional Hardy operator H_β and its adjoint operator H^*_β respectively, where

$$H_\beta f(x)=\frac{1}{|x|^{n-\beta}}\int_{|y|<|x|}f(y)dy,\quad x\in\mathbb{R}^n\backslash\{0\}\quad\text{and}\quad H^*_\beta f(x)=\int_{|y|\ge|x|}\frac{f(y)}{|y|^{n-\beta}}dy.$$