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Some Results on the Upper Convex Densities of the Self-Similar Sets at the Contracting-Similarity Fixed Points

Shaoyuan Xu[1,*], Wangbin Xu[2] and Zuoling Zhou[3]

[1] *School of Maths and Statistics, Hanshan Normal University, Chaozhou 521001, China*
[2] *School of Maths and Statistics, Hubei Normal University, Huangshi 435002, China*
[3] *School of Lingnan, Zhongshan University, Guangzhou 510275, China*

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Abstract. In this paper, some results on the upper convex densities of self-similar sets at the contracting-similarity fixed points are discussed. Firstly, a characterization of the upper convex densities of self-similar sets at the contracting-similarity fixed points is given. Next, under the strong separation open set condition, the existence of the best shape for the upper convex densities of self-similar sets at the contracting-similarity fixed points is proven. As consequences, an open problem and a conjecture, which were posed by Zhou and Xu, are answered.

Key Words: Self-similar set, upper convex density, Hausdorff measure and Hausdorff dimension, contracting-similarity fixed point.

AMS Subject Classifications: 28A78, 28A80

1 Introduction and preliminaries

It is well known that the theory of Hausdorff measure is the basis of fractal geometry and Hausdorff measure is an important notion in the study of fractals (see [1,2]). But unfortunately, it is usually difficult to calculate the exact value of the Hausdorff measures of fractal sets. Since Hutchinson [3] first introduced the self-similar set satisfying the open set condition (OSC), many authors have studied this class of fractals and obtained a number of meaningful results (see [1–10]). Among them, Zhou and Feng's paper [5] has attracted widespread attention since it was published in 2004. In [5], Zhou and Feng thought the reason for the difficulty in calculating Hausdorff measures of fractals is neither computational trickiness nor computational capacity, but a lack of full understanding of the essence of Hausdorff measure. Some authors recently studied self-similar sets

*Corresponding author. *Email address:* `xushaoyuan@126.com` (S. Y. Xu)

by means of upper convex density and best covering, which are very important to the study of Hausdorff measure (see [5–10]). In [5], Zhou and Feng posed eight open problems and six conjectures on Hausdorff measure of similar sets. Among them, a problem and a conjecture are as follows.

Let $E\subset R^n$ be a self-similar set satisfying OSC, the Hausdorff dimension of E be s, i.e., $\dim_H E=s$, and $x\in R^n$. Denote by $\overline{D}_C^s(E,x)$ the upper convex density of E at the point x.

Problem 1.1 (see [5])**.** Under what conditions is there a subset U_x in R^n with $|U_x|>0$ such that

$$\overline{D}_C^s(E,x)=\frac{H^s(E\cap U_x)}{|U_x|^s}?$$

Such a set U_x is called a best shape for the upper convex density of E at the point x.

Conjecture 1.1 (see [5])**.** $s=\dim_H E>1\Rightarrow$ there is an $x\in E$ such that $\overline{D}_C^s(E,x)<1$. Furthermore, $\overline{D}_C^s(C\times C,A)<1$, where $C\times C$ denotes the Cartesian product of the middle third Cantor set with itself and A denotes any vertex of $C\times C$ (see [5, Fig. 4]).

Recently, in order to study Conjecture 1.1 above, Xu [6] and Xu and Zhou [7] introduced the notion "contracting-similarity fixed point", and obtained a sufficient and necessary condition for the upper convex density of the self-similar s-set at the simple-contracting-similarity fixed point less than 1. In [7], a conjecture was posed as follows.

Conjecture 1.2 (see [7])**.** Let $E\subset R^n$ be a self-similar s-set satisfying OSC. Suppose that x is a contracting-similarity fixed point of E. Then $\overline{D}_C^s(E,x)<1$ if and only if $H^s(E\cap U)<|U|^s$ for each compact subset U in R^n with $x\in U$ and $|U|>0$.

This is an important conjecture connecting Hausdorff measure and upper convex density. In Xu [6], it was shown that Conjecture 1.2 would be true if we only considered the upper convex density at the simple-contracting-similarity fixed point of a self-similar s-set satisfying the strong separation set condition (SSC), instead of the one at the contracting-similarity fixed point of a self-similar s-set satisfying OSC. In the subsequent section (i.e., Section 2), we will set up a characterization of the upper convex densities of self-similar set at the contracting-similarity fixed points. Then, under the strong separation condition (SSC), we show that the existence of the best shape for the upper convex densities of the self-similar sets at the contracting-similarity fixed points. As applications, we answer an open problem (i.e., Problem 1.1 above), which was posed by Zhou and Feng in 2004. As consequences, we prove Conjecture 1.2 does hold true in the case that SSC is satisfied, thus generalizing the corresponding the result in Xu [6]. Some definitions, notations and known results are from references [1–4].

Let d be the standard distance function on R^n, where R^n is Euclidian n-space. Denote $d(x,y)$ by $|x-y|$, $\forall x,y\in R^n$. If U is a nonempty subset of R^n, we define the diameter of U as $|U|=\sup\{|x-y|:x,y\in U\}$. Let δ be a positive number. If $E\subset\bigcup_i U_i$ and $0<|U_i|\leq\delta$ for each i, we say that $\{U_i\}$ is a δ-covering of E.