Analysis in Theory and Applications
Anal. Theory Appl., Vol. **33**, No. 1 (2017), pp. 59-73

DOI: 10.4208/ata.2017.v33.n1.6

Analysis of Formal and Analytic Solutions for Singularities of the Vector Fractional Differential Equations

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Received 25 March 2014; Accepted (in revised version) 7 December 2016

Abstract. In this article, we study on the existence of solution for a singularities of a system of nonlinear fractional differential equations (FDE). We construct a formal power series solution for our considering FDE and prove convergence of formal solutions under conditions. We use the Caputo fractional differential operator and the nonlinearity depends on the fractional derivative of an unknown function.

Key Words: Fractional differential equations, formal power series solution.

AMS Subject Classifications: 26A33, 34B18, 34A08

1 Introduction

Recently, fractional differential equations have been investigated extensively. The motivation for those works rises from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, chemistry, aerodynamics, electrodynamics of complex medium, and so on. For examples and details, see [1–6, 8, 9, 11, 14, 16, 20, 23, 25, 26] and the references therein. Fractional calculus in the complex plane also was done by Osler and et al. [30–33]

The existence of formal and analytic solutions for singularities of ordinary differential equations such as $x\frac{d\vec{y}}{dx}=\vec{f}(x,\vec{y})$ and other statements was discussed in [27–29]. Motivated by the above mentioned work, in this paper we consider a system of singularities nonlinear fractional order differential equation:

$$x^{\alpha}\frac{d^{\alpha}\overrightarrow{y}}{dx^{\alpha}}=\overrightarrow{f}(x,\overrightarrow{y}), \quad \alpha\geq 1. \tag{1.1}$$

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The paper has been organised as follows. In Section 2 we give basic definitions and preliminary. Leibniz rule and chain rule for LFD have been derived in Section 3 and Section 4. Extensions of directional LFDs and local fractional Taylor series to higher orders have been presented in Sections 5 and 6.

2 Preliminaries

In this section, we present some notations, definitions and preliminary that will be useful for our main results. This materials can be found in the literatures [10, 17, 19, 21, 22, 24].

$$\vec{y}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \vec{f}=\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}, \quad |\vec{y}|=\max\{|y_1|,|y_2|,\cdots,|y_n|\}.$$

$\vec{f}(x,\vec{y})$ is $\mathbb{C}^n-$valued function in complex variables $(x,\vec{y})\in\mathbb{C}^{n+1}$. We denote by $\mathbb{C}[[x]]$ the set of all formal power series in x with coefficients in $\mathbb{C}$. Also, denote by $\mathbb{C}\{x\}$ the set of all power series in $\mathbb{C}[[x]]$ that have nonzero radii of convergence. Denote by $x^\sigma\mathbb{C}[[x]]$ the set of formal series $x^\sigma f(x)$, where $f(x)\in\mathbb{C}[[x]],\sigma$ is a complex number, and $x^\sigma=\exp(\sigma\ln x)$. Similarly, let $x^\sigma\mathbb{C}\{x\}$ denote the set of convergent series $x^\sigma f(x)$, where $f(x)\in\mathbb{C}\{x\}$.

Definition 2.1. A formal power series

$$\vec{\phi}(x)=\sum_{m=0}^{\infty}x^m\vec{c}_m\in x\mathbb{C}[[x]]^n, \quad \vec{c}_m\in\mathbb{C}^n,$$

is a formal solution of system (1.1) if

$$x^\alpha\frac{d^\alpha\vec{\phi}(x)}{dx^\alpha}=\vec{f}(x,\vec{\phi}(x)).$$

Riemanns modified form of Liouvilles fractional integral operator is a generalization of Cauchys iterated integral formula

$$\int_a^x dx_1\int_a^{x_1}dx_2\cdots\int_a^{x_{n-1}}g(x_n)dx_n=\frac{1}{\Gamma(n)}\int_a^x(x-s)^{n-1}g(s)ds, \tag{2.1}$$

where Γ is Euler's gamma function. Clearly, the right-hand side of Eq. (2.1) is meaningful for any positive real value of n. Hence, it is natural to define the fractional integral as follows:

Definition 2.2. If y be analitic function in $\mathbb{C}$, then the Riemann-Liouville fractional integral is defined by

$$\frac{d^{-\alpha}y(x)}{dx^\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}y(s)ds. \tag{2.2}$$