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TOPOLOGICAL ENTROPY AND IRREGULAR RECURRENCE

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Abstract. This paper is devoted to problems stated by Z. Zhou and F. Li in 2009. They concern relations between almost periodic, weakly almost periodic, and quasi-weakly almost periodic points of a continuous map f and its topological entropy. The negative answer follows by our recent paper. But for continuous maps of the interval and other more general one-dimensional spaces we give more results; in some cases the answer is positive.

Key words: *topological entropy, weakly almost periodic point, quasi-weakly almost periodic point*

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1 Introduction

Let (X,d) be a compact metric space, $I=[0,1]$ the unit interval, and $\mathcal{C}(X)$ the set of continuous maps $f:X\to X$. By $\omega(f,x)$ we denote the *ω-limit set* of x which is the set of limit points of the *trajectory* $\{f^i(x)\}_{i\geq 0}$ of x, where f^i denotes the ith iterate of f. We consider the sets $W(f)$ of *weakly almost periodic points* of f, and $QW(f)$ of *quasi-weakly almost periodic points* of f. They are defined as follows, see [11]:

$$W(f)=\left\{x\in X;\forall\varepsilon\ \exists N>0 \text{ such that } \sum_{i=0}^{nN-1}\chi_{B(x,\varepsilon)}(f^i(x))\geq n,\forall n>0\right\},$$

$$QW(f)=\left\{x\in X;\forall\varepsilon\ \exists N>0,\exists\{n_j\} \text{ such that } \sum_{i=0}^{n_jN-1}\chi_{B(x,\varepsilon)}(f^i(x))\geq n_j,\forall j>0\right\},$$

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where $B(x,\varepsilon)$ is the ε-neighbourhood of x, χ_A the characteristic function of a set A, and $\{n_j\}$ an increasing sequence of positive integers. For $x \in X$ and $t > 0$, let

$$\Psi_x(f,t) = \liminf_{n\to\infty} \tfrac{1}{n}\#\{0 \le j < n; d(x, f^j(x)) < t\}, \tag{1}$$

$$\Psi_x^*(f,t) = \limsup_{n\to\infty} \tfrac{1}{n}\#\{0 \le j < n; d(x, f^j(x)) < t\}. \tag{2}$$

Thus, $\Psi_x(f,t)$ and $\Psi_x^*(f,t)$ are the *lower* and *upper Banach density* of the set $\{n \in \mathbf{N}; f^n(x) \in B(x,t)\}$, respectively. In this paper we make of use more convenient definitions of $W(f)$ and $QW(f)$ based on the following lemma.

Lemma 1. *Lef $f \in \mathcal{C}(X)$. Then*

(i) *$x \in W(f)$ if and only if $\Psi_x(f,t) > 0$, for every $t > 0$,*

(ii) *$x \in QW(f)$ if and only if $\Psi_x^*(f,t) > 0$, for every $t > 0$.*

Proof. It is easy to see that, for every $\varepsilon > 0$ and $N > 0$,

$$\sum_{i=0}^{nN-1} \chi_{B(x,\varepsilon)}(f^i(x)) \ge n \quad \text{if and only if} \quad \#\{0 \le j < nN; f^j(x) \in B(x,\varepsilon)\} \ge n. \tag{3}$$

(i) If $x \in W(f)$ then, for every $\varepsilon > 0$ there is an $N > 0$ such that the condition on the left side in (3) is satisfied for every n. Hence, by the condition on the right, $\Psi_x(f,\varepsilon) \ge 1/N > 0$. If $x \notin W(f)$ then there is an $\varepsilon > 0$ such that for every $N > 0$, there is an $n > 0$ such that the condition on the left side of (3) is not satisfied. Hence, by the condition on the right, $\Psi_x(f,t) < 1/N \to 0$ if $N \to \infty$. Proof of (ii) is similar.

Obviously, $W(f) \subseteq QW(f)$. The properties of $W(f)$ and $QW(f)$ were studied in the nineties by Z. Zhou et al, see [11] for references. The points in $IR(f) := QW(f) \setminus W(f)$ are *irregularly recurrent points*, i.e., the points x such that $\Psi_x^*(f,t) > 0$ for any $t > 0$, and $\Psi_x(f,t_0) = 0$ for *some* $t_0 > 0$, see [7]. Denote by $h(f)$ the *topological entropy* of f and by $R(f)$, $UR(f)$ and $AP(f)$ the set of *recurrent*, *uniformly recurrent* and *almost periodic* points of f, respectively. Thus, $x \in R(f)$ if for every neighborhood U of x, $f^j(x) \in U$ for infinitely many $j \in \mathbf{N}$; $x \in UR(f)$ if for every neighborhood U of x there is a $K > 0$ such that every interval $[n, n+K]$ contains a $j \in \mathbf{N}$ with $f^j(x) \in U$; and $x \in AP(f)$ if for every neighborhood U of x, there is a $k > 0$ such that $f^{kj}(x) \in U$ for every $j \in \mathbf{N}$. Recall that $x \in R(f)$ if and only if $x \in \omega(f,x)$, and $x \in UR(f)$ if and only if $\omega(f,x)$ is a *minimal set*, i.e., a closed set $\emptyset \ne M \subseteq X$ such that $f(M) = M$ and no proper subset of M has this property. Denote by $\omega(f)$ the union of all ω-limit sets of f. The next relations follow by definition:

$$AP(f) \subseteq UR(f) \subseteq W(f) \subseteq QW(f) \subseteq R(f) \subseteq \omega(f) \tag{4}$$