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REGULARIZATION OF AN ILL-POSED HYPERBOLIC PROBLEM AND THE UNIQUENESS OF THE SOLUTION BY A WAVELET GALERKIN METHOD

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Abstract. We consider the problem $K(x)u_{xx} = u_{tt}$, $0 < x < 1$, $t \geq 0$, with the boundary condition $u(0,t) = g(t) \in L^2(\mathbf{R})$ and $u_x(0,t) = 0$, where $K(x)$ is continuous and $0 < \alpha \leq K(x) < +\infty$. This is an ill-posed problem in the sense that, if the solution exists, it does not depend continuously on g. Considering the existence of a solution $u(x,\cdot) \in H^2(\mathbf{R})$ and using a wavelet Galerkin method with Meyer multiresolution analysis, we regularize the ill-posedness of the problem. Furthermore we prove the uniqueness of the solution for this problem.

Key words: *ill-posed problem, meyer wavelet, hyperbolic equation, regularization*

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1 Introduction and Main Results

In [5] the authors have considered an inverse problem for the sideway heat equation with constant coefficient. The variational formulation, on the scaling space V_j, of the approximating problem, produces an infinite-dimensional system of second order ordinary differential equations with constant coefficients, for which the solution is known. Stability and convergence of the method follow the from form of this solution.

In a previous work[3], we studied the following parabolic partial differential equation problem with variable coefficients:

$$\begin{aligned} K(x)u_{xx}(x,t) &= u_t(x,t), & t \geq 0, \quad 0 < x < 1 \\ u(0,\cdot) &= g, & u_x(0,\cdot) = 0 \\ 0 < \alpha \leq K(x) &< +\infty, & K \quad \text{continuous}. \end{aligned}$$

Under the hypothesis of the existence of a solution for this problem, using a wavelet Galerkin method, we constructed a sequence of well-posed approximating problems in the scaling spaces of the Meyer multiresolution analysis, which has the property to filter away the high frequencies. We had shown the convergence of the method, applied to our problem, and we gave an estimate of the solution error. We get an estimate for the difference between the exact solution of this problem and the orthogonal projection, onto V_j, of the solution of the approximating problem defined on the scaling space V_{j-1}.

In [6] the authors have given the error estimate between the exact solution by the above problem and the approximating solution of wavelet-Galerkin method in the sense of pointwise convergence.

In our work[12], by assuming that $\dfrac{1}{K(x)}$ is Lipschitz, we proved that the existence of a solution $u(x,.) \in H^1(\mathbf{R})$, for the above problem, implies its uniqueness.

In this work, we will extend the results in [2] and [3] to the hyperbolic problem:

$$\begin{gathered} K(x)u_{xx}(x,t) = u_{tt}(x,t), \quad t \geq 0,\ 0 < x < 1 \\ u(0,\cdot) = g, \quad u_x(0,\cdot) = 0 \\ 0 < \alpha \leq K(x) < +\infty, \quad K \ \text{continuous}. \end{gathered} \tag{1.1}$$

We assume $g \in L^2(\mathbf{R})$, when it is extended as vanishing for $t < 0$, and the problem to have a solution $u(x,\cdot) \in H^2(\mathbf{R})$, when it is extended as vanishing for $t < 0$.

Our approach follows quite closely to that used in [2] and [3].

In note 1 we show that problem (1.1) is ill-posed in the sense that a small disturbance on the boundary specification g, can produce a big alteration on its solution, if it exists.

We consider the Meyer multiresolution analysis. The advantage in making use of Meyer's wavelets is its good localization in the frequency domain, since its Fourier transform has compact support. Orthogonal projections onto Meyer's scaling spaces, can be considered as low pass filters, cutting off the high frequencies.

From the variational formulation of the approximating problem on the scaling space V_j, we get an infinite-dimensional system of second order ordinary differential equations with variable coefficients. An estimate obtained for the solution of this evolution problem, is used to regularize the ill-posed problem approaching it by well-posed problems. Using an estimate obtained for the difference between the exact solution of problem (1.1) and its orthogonal projection onto