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NOTE ON THE PAPER " AN NEGATIVE ANSWER TO A CONJECTURE ON THE SELF-SIMILAR SETS SATISFYING THE OPEN SET CONDITION"

Shaoyuan Xu

(*Gannan Normal University, China*)

and

Wangbin Xu

(*Hubei Normal University, China)*

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Abstract. In this paper, we present a more simple and much shorter proof for the main result in the paper " An negative answer to a conjecture on the self-similar sets satisfying the open set condition", which was published in the journal Analysis in Theory and Applications in 2009.

Key words: *self-similar set, best covering, natural covering, Hausdorff measure and Hausdorff dimension*

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1 Introduction

It is well known that the theory of Hausdorff measure is the basis of fractal geometry and Hausdorff measure is an important notion in the study of fractals (see [1-2]). But unfortunately, it is usually difficult to calculate the exact value of the Hausdorff measure of fractal sets. Since J.E.Hutchinson [3] first introduced self-similar sets satisfying the open set condition (OSC), many authors have studied this class of fractals and obtained a number of meaningful results (see [1-11]). Among them, Z. Zhou and L. Feng's paper[5] has attracted widespread attention since it was published in 2004. In [5], Z. Zhou and L. Feng thought the reason for the difficulty

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in calculating the Hausdorff measures of fractals is neither computational trickiness nor computational capacity, but a lack of full understanding of the essence of the Hausdorff measure. They posed eight open problems and six conjectures on Hausdorff measure of similar sets. Among them is a conjecture as follows:

Conjecture 1.1[5]. *E has a best covering if and only if* $H^s(E) = |E|^s$.

Recently, some authors negatively answered the above-mentioned Conjecture 1.1 (see [8, 9]). In [8], the authors neglected an important condition that a best covering $\{U_i\}_{i\in I}$ must meet, that is, for any $i \in I$, the inequality $|U_i| > 0$ should hold. They constructed a self-similar set which is for the IFS consisting of three functions, found a covering $\beta = \{\{1\}, [0, \frac{3}{9}], \cdots\}$ of E and claimed that it was a best covering of E. In fact, this covering is an best almost covering but not a best one since the diameter of the set $\{1\}$ is 0. In [9], the author constructed another self-similar set which is for the IFS consisting of six functions and claimed that by discussing this similar set, the main result (i.e., Theorem 3.1 in [9]) can provide a negative answer to the conjecture, but unfortunately the proof is too long. In this paper, we will present a more simple and much shorter shirt proof of the result. The objective of this paper is to promote more people to develop interest in this conjecture and its answer.

Some definitions, notations and known results are from references [1-5].

Let d be the standard distance function on R^n, where R^n is Euclidian n-space. Denote $d(x,y)$ by $|x-y|$, $\forall x,y \in R^n$. If U is a nonempty subset of R^n, we define the diameter of U as $|U| = \sup\{|x-y| : x,y \in U\}$. Let δ be a positive number. If $E \subset \bigcup_i U_i$ and $0 < |U_i| \le \delta$ for each i, we say that $\{U_i\}$ is a δ-covering of E.

Let $E \subset R^n$ and $s \ge 0$. For $\delta > 0$, define

$$H^s_\delta(E) = \inf\{\sum_i |U_i|^s : \bigcup_i U_i \supset E, 0 < |U_i| \le \delta\}.$$

Letting $\delta \to 0$, we call the limit

$$H^s(E) = \lim_{\delta\to 0} H^s_\delta(E)$$

the s-dimensional Hausdorff measure of E. Note that the Hausdorff dimension of E is defined as

$$\dim_H E = \inf\{s \ge 0 : H^s(E) = 0\} = \sup\{s \ge 0 : H^s(E) = \infty\}.$$

An H^s-measurable set $E \subset R^n$ with $0 < H^s(E) < \infty$ is termed an s-set.

Now we review the self-similar s-sets satisfying the open set condition. Let $D \subset R^n$ be closed. A map $S : D \to D$ is called a contracting similarity, if there is a number c with $0 < c < 1$ such that

$$|S(x) - S(y)| = c|x-y|, \qquad \forall\, x,y \in D$$