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ESTIMATES OF LINEAR RELATIVE n-WIDTHS IN $L^p[0,1]$

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Abstract. In this paper we will show that if an approximation process $\{L_n\}_{n\in\mathbf{N}}$ is shape-preserving relative to the cone of all k-times differentiable functions with non-negative k-th derivative on [0,1], and the operators L_n are assumed to be of finite rank n, then the order of convergence of D^kL_nf to D^kf cannot be better than n^{-2} even for the functions x^k, x^{k+1}, x^{k+2} on any subset of [0,1] with positive measure. Taking into account this fact, we will be able to find some asymptotic estimates of linear relative n-width of sets of differentiable functions in the space $L^p[0,1]$, $p\in\mathbf{N}$.

Key words: *shape preserving approximation, linear n-width*

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1 Introduction

In various applications it is necessary to approximate functions preserving their properties such as monotonicity, convexity, concavity, etc. The last 25 years have seen extensive research in the theory of shape preserving approximation by means of polynomials and splines. The most significant results can be found in [1], [2].

Note that if a function f has some shape properties, it usually means that the element f belongs to a cone V in $C[0,1]$. A linear operator L defined in $C[0,1]$ is said to be shape-preserving relative to the cone V, if $L(V)\subset V$.

One of the most examined classes of linear shape-preserving operators is the class of linear positive operators. It is well-known that one of the shortcomings of linear positive operators is their slow convergence. It was shown by P. P. Korovkin[3] that the order of approximation by positive linear polynomial operators of degree n can not be better than n^{-2} in $C[0,1]$ even for the functions 1, x, x^2. Moreover, V. S. Videnskii[4] has shown that the result of [3] does not depend on the properties of polynomials but rather on the limitation of dimension.

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A function $f:[0,1]\to \mathbf{R}$ is said to be p-convex, $p\geq 1$, on $[0,1]$ iff for all choices of $p+1$ distinct $t_0,\cdots,t_p$ in $[0,1]$ the inequality

$$[t_0,\cdots,t_p]f\geq 0$$

holds, where

$$[t_0,\cdots,t_p]f=\sum_{j=0}^{p}(f(t_j)/w'(t_j))$$

denotes the p-th divided difference of f at $0\leq t_0<t_1<\cdots<t_p\leq 1$, and $w(t)=\prod_{j=0}^{p}(t-t_j)$.

Note that 2-convex functions are just convex functions. The class of all p-convex functions on $[0,1]$ is denoted by $\Delta^p[0,1]$. If $f\in C^p[0,1]$, then $f\in\Delta^p[0,1]$ iff $f^{(p)}(t)\geq 0$, $t\in[0,1]$. Let $\Delta^0[0,1]:=\{f\in C[0,1]:\ f(t)\geq 0,\ t\in[0,1]\}$.

Let $0\leq h\leq k$ be two integers and let $\sigma=(\sigma_h,\ldots,\sigma_k)\in R^{k-h+1}$, $\sigma_i\in\{-1,0,1\}$ be such that $\sigma_h\sigma_k\neq 0$.

Following the idea of [5] denote

$$\Delta^{h,k}(\sigma):=\{f\in C[0,1]:\ \sigma_i f\in\Delta^i[0,1],\ h\leq i\leq k\}.$$

We will use the notations $\sigma^{[k]}=\left(\sigma_i^{[k]}\right)_{i=h}^{k}$ with $\sigma_i^{[k]}=0$ for $i\neq k$ and $\sigma_k^{[k]}=\sigma_k$.

A linear operator mapping $C[0,1]$ into a linear space of finite dimension n is called an operator of finite rank n. Let D^k denote the k-th differential operator, $D^k(x):=\mathrm{d}^k f(x)/\mathrm{d}x^k$.

Denote $e_j(t)=t^j$, $j=0,1,\cdots$. Let $\Omega\subset[0,1]$ be a measurable set with $\mathrm{meas}(\Omega)\neq 0$. Let $\{L_n\}_{n\in\mathbf{N}}$, $L_n:C^k[0,1]\to C^k(\Omega)$, be a sequence of linear shape-preserving operators, such that for all $n\in\mathbf{N}$ we have $L_n(\Delta^{h,k}(\sigma))\subset\Delta^{h,k}(\sigma^{[k]})$ and L_n is of finite rank n. Extending the results of Korovkin[3], of Videnskii[4], of Vasiliev and Guendouz [6], in this paper we will show that the measure of the set of all $x\in\Omega$, such that

$$\lim_{n\to\infty} n^2|D^kL_ne_j(x)-D^ke_j(x)|=0,\qquad j=k,\ k+1,\ k+2,$$

is equal to zero. Using this fact, we will find asymptotic estimates of linear relative n-width of sets of differentiable functions in the space $L^p[0,1]$.

2 The Order of Approximation by Means of Linear Shape-Preserving Operators

Let $E\subset[0,1]$ be a closed set with $\mathrm{meas}(E)>0$. Given $m\in\mathbf{N}$, denote $X=\{x_i\}$, where

$$x_i=\frac{i}{m},\qquad i=-1,0,1,\cdots,m+1.$$