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# Square Root Functional Equation on Positive Cones

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**Abstract.** A square root functional equation on positive cones of $C^*$-algebras is introduced and its solution and Hyers-Ulam-Rassias stability are investigated.

**Key Words**: Hyers-Ulamstability, fixed point, additive functional equation.

**AMS Subject Classifications**: 39B52, 47H10

## 1 Introduction

The stability theory of functional equation is originated from the well-known Ulam's problem [1] concerning the stability of homomorphisms in metric groups: Let $(G,*)$ be a group and $(X,\cdot)$ be a metric group. Does for every $\varepsilon>0$ there exist $\delta>0$ such that if $f:G\to X$ satisfies

$$d(f(x*y),f(x)\cdot f(y))<\delta \quad \text{for } x,y\in G,$$

then a homomorphism $h:G\to X$ exists with $d(f(x),h(x))<\varepsilon$ for $x\in G$? Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers's Theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference.

**Theorem 1.1** (Th. M. Rassias)**.** *Consider two Banach spaces $E_1$, $E_2$, and let $f:E_1\to E_2$ be a mapping such that $f(tx)$ is continuous in $t$ for each fixed $x$. Assume that there exist $\theta\ge 0$ and $p\in[0,1)$, such that*

$$\frac{\|f(x+y)-f(x)-f(y)\|}{\|x\|^p+\|y\|^p}\le\theta \quad \text{for any } x,y\in E_1.$$

*Then there exists a unique linear mapping $T:E_1\to E_2$ such that*

$$\frac{\|f(x)-T(x)\|}{\|x\|^p}\le\frac{2\theta}{2-2^p} \quad \text{for any } x\in E_1.$$

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The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Gavruta [5] by using a general controb function in place of the unbounded Cauchy difference in the spirit of Th. M. Rassias's approach. Following the innovative approach of the Th. M. Rassias theorem [4], J. M. Rassias [6] replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\|y\|^q$ for $p,q\in\mathbb{R}$ with $p+q=1$. The stability problem of several functional equations has been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7–10]). Let $\mathcal{A}$ be a $C^*$-algebra and $a\in\mathcal{A}$ be a self-adjoint element, i.e., $a=a^*$. Then $a$ is said to be positive if it is of the form $a=bb^*$ for some $a\in\mathcal{A}$. The set of positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}^+$. Note that $\mathcal{A}^+$ is a closed convex cone (see [11]). Moreover, It is well-known that for a positive element $a$ and a positive integer $n$ there exists a unique positive element $x\in\mathcal{A}^+$ such that $a=x^n$. In this case, we denote $x$ by $\sqrt[n]{a}$. In the following some preliminary properties of $\mathcal{A}^+$ are listed [11]:

**Theorem 1.2.** *Suppose that $\mathcal{A}$ is a $C^*$-algebra.*

*(i)* $\mathcal{A}^+$ *is closed in* $\mathcal{A}$,

*(ii)* $ax\in\mathcal{A}^+$ *if* $x\in\mathcal{A}^+$ *and* $a\geq 0$,

*(iii)* $x+y\in\mathcal{A}^+$ *if* $x,y\in\mathcal{A}^+$,

*(iv)* $xy\in\mathcal{A}^+$ *if* $x,y\in\mathcal{A}^+$ *and* $xy=yx$,

*(v)* $x\in\mathcal{A}^+$ *and* $-x\in\mathcal{A}^+$, *then* $x=0$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$-algebra and $\mathcal{A}^+$ and $\mathcal{B}^+$ be the corresponding positive cones. We introduce the following pair of functional equations

$$\begin{cases} f(x)f(y)=f(y)f(x), \\ f(ax+by)=a^2f(x)+2ab\sqrt{f(x)f(y)}+b^2f(y), \end{cases} \tag{1.1}$$

for every $x,y\in\mathcal{A}^+$ and $f:\mathcal{A}^+\to\mathcal{B}^+$. Here $a$, $b$ are two nonnegative real scalars that $a+b\neq 0$. By part (iv) of Theorem 1.2, the first equation of condition (1.1) is needed for the second equation of (1.1) to be well-defined. Note that the function $f:\mathcal{A}^+\to\mathcal{B}^+$ by $f(x)=cx^2$, $c\geq 0$, is a solution of the functional equation (1.1). Applying (1.1) for $x=y=0$, $x=0$, and $y=0$, separately, gives $f(0)=0$; $f(ax)=a^2f(x)$, and $f(by)=b^2f(y)$, respectively. Hence (1.1) can be modified by the following:

$$f(ax+by)=f(ax)+2\sqrt{f(ax)f(by)}+f(by)=\Big(\sqrt{f(ax)}+\sqrt{f(by)}\Big)^2,$$

and consequently,

$$\sqrt{f(ax+by)}=\sqrt{f(ax)}+\sqrt{f(by)}.$$