Analysis in Theory and Applications
Anal. Theory Appl., Vol. **29**, No. 1 (2013), pp. 72-85
DOI: 10.4208/ata.2013.v29.n1.8

Some Estimates for Commutators of Fractional Integrals Associated to Operators with Gaussian Kernel Bounds on Weighted Morrey Spaces

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Received 22 October 2012

Abstract. Let L be the infinitesimal generator of an analytic semigroup on $L^2(\mathbf{R}^n)$ with Gaussian kernel bound, and let $L^{-\alpha/2}$ be the fractional integrals of L for $0<\alpha<n$. In this paper, we will obtain some boundedness properties of commutators $[b,L^{-\alpha/2}]$ on weighted Morrey spaces $L^{p,\kappa}(w)$ when the symbol b belongs to $BMO(\mathbf{R}^n)$ or the homogeneous Lipschitz space.

Key Words: Gaussian upper bound, fractional integral, weighted Morrey space, commutator.

AMS Subject Classifications: 42B20, 42B35

1 Introduction

Suppose that L is the infinitesimal generator of an analytic semigroup $\{e^{-tL}\}_{t>0}$ on $L^2(\mathbf{R}^n)$ with a kernel $p_t(x,y)$ satisfying a Gaussian upper bound; that is, there exist positive constants C and A such that for all $x,y\in\mathbf{R}^n$ and all $t>0$, we have

$$|p_t(x,y)|\le\frac{C}{t^{n/2}}e^{-A\frac{|x-y|^2}{t}}. \tag{1.1}$$

Throughout this paper, we assume that the semigroup $\{e^{-tL}\}_{t>0}$ has a kernel satisfying (1.1). This property is satisfied by a large class of differential operators, as is seen in [7].

For any $0<\alpha<n$, the fractional integral $L^{-\alpha/2}$ associated to the operator L is defined by

$$L^{-\alpha/2}f(x)=\frac{1}{\Gamma(\alpha/2)}\int_0^\infty e^{-tL}(f)(x)t^{\alpha/2-1}dt. \tag{1.2}$$

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Note that if $L=-\Delta$ is the Laplacian on $\mathbf{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integral operator I_α, which is given by (see [20])

$$I_\alpha f(x)=\frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}\int_{\mathbf{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

Let b be a locally integrable function on $\mathbf{R}^n$. The commutator of b and $L^{-\alpha/2}$ is defined as follows

$$\big[b,L^{-\alpha/2}\big](f)(x)=b(x)L^{-\alpha/2}(f)(x)-L^{-\alpha/2}(bf)(x). \tag{1.3}$$

The first result on the theory of commutators is obtained by Coifman, Rochberg and Weiss in [3]. Since then, many authors have been interested in studying this theory. When $0<\alpha<n$, $1<p<n/\alpha$ and $1/q=1/p-\alpha/n$, Chanillo [2] proved that the commutator $[b,I_\alpha]$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ whenever $b\in BMO(\mathbf{R}^n)$. Paluszyński [18] showed that $b\in\dot{\Lambda}_\beta(\mathbf{R}^n)$ (homogeneous Lipschitz space) if and only if $[b,I_\alpha]$ is bounded from $L^p(\mathbf{R}^n)$ to $L^s(\mathbf{R}^n)$, where $0<\beta<1$, $1<p<n/(\alpha+\beta)$ and $1/s=1/p-(\alpha+\beta)/n$. For the weighted case, Segovia and Torrea [19] proved that when $b\in BMO(\mathbf{R}^n)$ and $w\in A_{p,q}$ (Muckenhoupt weight class), $[b,I_\alpha]$ is bounded from $L^p(w^p)$ to $L^q(w^q)$.

In 2004, by using a new sharp maximal function introduced in [14], Duong and Yan [7] extended the result of [2] from $(-\Delta)$ to the more general operator L defined above. More precisely, they showed that

Theorem 1.1. *Let $0<\alpha<n$, $1<p<n/\alpha$ and $1/q=1/p-\alpha/n$. If $b\in BMO(\mathbf{R}^n)$, then the commutator $\big[b,L^{-\alpha/2}\big]$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$.*

In 2008, Auscher and Martell [1] considered the weighted case and obtained the following result (see also [4]).

Theorem 1.2. *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $w\in A_{p,q}$. If $b\in BMO(\mathbf{R}^n)$, then the commutator $\big[b,L^{-\alpha/2}\big]$ is bounded from $L^p(w^p)$ to $L^q(w^q)$.*

On the other hand, in 2009, Komori and Shirai [13] first introduced the weighted Morrey spaces $L^{p,\kappa}(w)$ which could be viewed as an extension of weighted Lebesgue spaces, and investigated the boundedness of the Hardy-Littlewood maximal operator, singular integral operator and fractional integral operator on these weighted spaces. Moreover, they also proved the following theorem.

Theorem 1.3. *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $0<\kappa<p/q$ and $w\in A_{p,q}$. If $b\in BMO(\mathbf{R}^n)$, then the commutator $[b,I_\alpha]$ is bounded from $L^{p,\kappa}(w^p,w^q)$ to $L^{q,\kappa q/p}(w^q)$.*

The purpose of this paper is to study the boundedness of $\big[b,L^{-\alpha/2}\big]$ on the weighted Morrey spaces $L^{p,\kappa}(w)$ when $b\in BMO(\mathbf{R}^n)$ or $b\in\dot{\Lambda}_\beta(\mathbf{R}^n)$. Our main results are formulated as follows.