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Inequalities for the Polar Derivatives of a Polynomial

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Abstract. Let $P(z)$ be a polynomial of degree n, having all its zeros in $|z|\le 1$. In this paper, we estimate kth polar derivative of $P(z)$ on $|z|=1$ and thereby obtain compact generalizations of some known results which among other things yields a refinement of a result due to Paul Tura'n.

Key Words: Polar derivative of a polynomial, maximum modulus, Bernstiens inequality.

AMS Subject Classifications: 30C10, 30D15

1 Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and C denote the complex plane, then concerning the estimate of $P'(z)$ on the unit circle $|z|=1$, we have the following famous result known as the Bernstien's inequality (for reference see [6])

$$\max_{|z|=1}|P'(z)|\le \max_{|z|=1}|P(z)|. \tag{1.1}$$

Equality holds in (1.1), if and only if $P(z)$ has all its zeros at the origin.

It was shown by Turan [8] that if $P(z)$ has all its zeros in $|z|\le 1$, then

$$\max_{|z|=1}|P'(z)|\ge \max_{|z|=1}|P(z)|. \tag{1.2}$$

Equality holds in (1.2) for those polynomials of degree n having all zeros on $|z|=1$.

Aziz and Dawood [1] obtained the following refinement of inequality (1.2) and proved that if $P(z)$ has all its zeros in $|z|\le 1$, then

$$\max_{|z|=1}|P'(z)|\ge \frac{n}{2}\Big[\max_{|z|=1}|P(z)|+\min_{|z|=1}|P(z)|\Big]. \tag{1.3}$$

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The result is best possible and equality holds for a polynomial having all zeros on $|z|=1$.

Let $D_\alpha P(z)$ denote the polar derivative of the polynomial $P(z)$ of degree n with respect to $\alpha\in C$, then

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalises the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\Big|\frac{D_\alpha P(z)}{\alpha}\Big|=P'(z).$$

Aziz and Rather [2] extended (1.3) to the polar derivative of $P(z)$ and proved the following result:

Theorem 1.1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\le 1$, then for all real and complex number α with $|\alpha|\ge 1$,*

$$\max_{|z|=1}|D_\alpha P(z)|\ge\frac{n}{2}\Big\{(|\alpha-1|)\max_{|z|=1}|P(z)|+(|\alpha|+1)\min_{|z|=1}|P(z)|\Big\}. \tag{1.4}$$

The result is best possible and equality in (1.4) holds for the polynomial

$$P(z)=(z-1)^n \quad \text{with } |\alpha|\ge 1.$$

As a generalisation of Theorem 1.1, Jain [4] proved the following result:

Theorem 1.2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\le 1$, then for $\alpha_1,\alpha_2,\cdots,\alpha_k\in C$, with $|\alpha_i|\ge 1$, $i=1,2,\cdots,k$ $(1\le k<n)$,*

$$\begin{aligned}&\max_{|z|=1}|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|\\ \ge&\frac{n(n-1)\cdots(n-k+1)}{2^k}\{(|\alpha_1|-1)(|\alpha_2|-1)\cdots(|\alpha_k|-1)\}\max_{|z|=1}|P(z)|\\ &+\Big(2^k\Big\{(|\alpha_1||\alpha_2|\cdots|\alpha_k|)\Big\}-(|\alpha_1|-1)(|\alpha_2|-1)\cdots(|\alpha_k|-1)\Big)\min_{|z|=1}|P(z)|.\end{aligned} \tag{1.5}$$

Where

$$D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)=P_j(z)=(n-j+1)P_{j-1}(z)+(\alpha_j-z)P'_{j-1}(z),\quad j=1,2,\cdots,k,$$
$$P_0(z)=P(0).$$

The result is best possible and equality holds for $P(z)=(z-1)^n$, with $|\alpha_i|\ge 1$, $i=1,2,\cdots,k$.

The aim of this paper is to present some interesting generalisations of Theorems 1.1 and 1.2 which among other things yields some known results as well. We first prove: