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Subordination Results for p-Valent Meromorphic Functions Associated with a Linear Operator

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Abstract. In this paper, by making use of the Hadamard products, we obtain some subordination results for certain family of meromorphic functions defined by using a new linear operator.

Key Words: Meromorphic functions, subordination, Hadamard product, linear operator.

AMS Subject Classifications: 30C45

1 Introduction

Let $\sum_p$ be the class of functions of the form:

$$f(z)=z^{-p}+\sum_{k=1-p}^{\infty}a_kz^k,\quad p\in\mathbb{N}=\{1,2,\cdots\},\tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^*=U\backslash\{0\}$, where $U=\{z: z\in\mathbb{C},\ |z|<1\}$. If f and g are analytic functions in U, we say that f is subordinate to g, written $f\prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ for all $z\in U$, such that $f(z)=g(w(z))$, $z\in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (cf. e.g., [6,7] and [3]):

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0)\quad\text{and}\quad f(U)\subset g(U).$$

For functions f given by (1.1) and $g\in\sum_p$ given by

$$g(z)=z^{-p}+\sum_{k=1-p}^{\infty}b_kz^k,$$

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the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^{-p}+\sum_{k=1-p}^{\infty}a_kb_kz^k=(g*f)(z).$$

For complex numbers $\alpha_1,\cdots,\alpha_q$ and $\beta_1,\cdots,\beta_s$ ($\beta_j\notin Z_0^-=\{0,-1,-2,\cdots\}$; $j=1,2,\cdots,s$), we define the generalized hypergeometric function ${}_qF_s$ $(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_s;z)$ (see, for example, [9]) by the following infinite series:

$${}_qF_s(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_s;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_q)_k}{(\beta_1)_k\cdots(\beta_s)_k(1)_k}z^k,\quad q\le s+1;\ s,q\in\mathbb{N}_0=\mathbb{N}\cup\{0\};\ z\in U,$$

where

$$(d)_k=\begin{cases}1, & k=0;\quad d\in\mathbb{C}^*=\mathbb{C}\backslash\{0\},\\ d(d+1)\cdots(d+k-1), & k\in\mathbb{N}_0;\quad d\in\mathbb{C}.\end{cases}$$

Corresponding to a function $K_{p,q,s}(\alpha_1;\beta_1;z)$ defined by

$$K_{p,q,s}(\alpha_1;\beta_1;z)=z_q^{-p}F_s(\alpha_1,\cdots,\alpha_q;\beta_1,\cdots,\beta_s;z), \tag{1.2}$$

we define the linear operator $H_{p,q,s}^{m,\lambda}(\alpha_1)f:\sum_p\rightarrow\sum_p$ by:

$$\begin{aligned}&H_{p,q,s}^{0,\lambda}(\alpha_1)f(z)=f(z)*K_{p,q,s}(\alpha_1;\beta_1;z),\\&H_{p,q,s}^{1,\lambda}(\alpha_1)f(z)=H_{p,q,s}^{\lambda}(\alpha_1)f(z)\\&=(1-\lambda)(f(z)*K_{p,q,s}(\alpha_1;\beta_1;z))+\frac{\lambda}{z^p}[z^{p+1}f(z)*K_{p,q,s}(\alpha_1;\beta_1;z)]',\end{aligned}$$

and (in general)

$$\begin{aligned}&H_{p,q,s}^{m,\lambda}(\alpha_1)f=H_{p,q,s}^{\lambda}(\alpha_1)(H_{p,q,s}^{m-1,\lambda}(\alpha_1)f(z))\\&=z^{-p}+\sum_{k=1-p}^{\infty}[1+\lambda(k+p)]^m\frac{(\alpha_1)_{k+p}\cdots(\alpha_q)_{k+p}}{(\beta_1)_{k+p}\cdots(\beta_s)_{k+p}(1)_{k+p}}a_kz^k,\quad m\in\mathbb{N}_0;\quad\lambda>0.\end{aligned}\tag{1.3}$$

From (1.3), we can easily deduce that

$$z(H_{p,q,s}^{m,\lambda}(\alpha_1)f(z))'=\alpha_1H_{p,q,s}^{m,\lambda}(\alpha_1+1)f(z)-(\alpha_1+p)H_{p,q,s}^{m,\lambda}(\alpha_1)f(z), \tag{1.4}$$

and

$$\lambda z(H_{p,q,s}^{m,\lambda}(\alpha_1)f(z))'=H_{p,q,s}^{m+1,\lambda}(\alpha_1)f(z)-(1+\lambda p)H_{p,q,s}^{m,\lambda}(\alpha_1)f(z),\quad\lambda>0. \tag{1.5}$$

We note that, for $m=0$, the operator $H_{p,q,s}^{0,\lambda}(\alpha_1)=H_{p,q,s}(\alpha_1)$ which was investigated by Liu and Srivastava [5] and Aouf [2], for $q=2$, $s=1$, $\alpha_2=1$ and $m=0$, the operator $H_{p,2,1}^{0,\lambda}(\alpha_1;\beta_1)=$