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# Hölder Estimate of Harmonic Functions on a Class of p.c.f. Self-Similar Sets

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**Abstract.** In this paper we establish sharp Hölder estimates of harmonic functions on a class of connected post critically finite (p.c.f.) self-similar sets, and show that functions in the domain of Laplacian enjoy the same property. Some well-known examples, such as the Sierpinski gasket, the unit interval, the level 3 Sierpinski gasket, the hexagasket, the 3-dimensional Sierpinski gasket, and the Vicsek set are also considered.

**Key Words**: p.c.f. self-similar sets, Hölder estimates, harmonic function.

**AMS Subject Classifications**: 28A80

## 1 Introduction

Hölder estimates play an important role in the theory of function spaces. Kigami used a constructive limit-of-difference-quotients method to define the harmonic function, Dirichlet form and the Laplacian on p.c.f. self-similar sets in [7–9]. His theory can be used to compute values of junction points for harmonic functions.

For p.c.f. self-similar sets there is an almost trivial weak bound on the Hölder continuity of harmonic functions and those in the domain of the Dirichlet form, which gives a useful estimate in terms of the resistance metric for most purposes. It is of course natural to ask for sharper results. For nested fractals such sharp results went back to Kumagai in [10]. Later in [4] Fitzsimmons, Hambly, and Kumgai extended the results for affine nested fractals. Barlow gave the estimate for the functions in the domain of Dirichlet form on p.c.f. self-similar sets in [2].

Strichartz has established a Hölder estimate of harmonic functions on the Sierpinski gasket in [12]. Then a Hölder estimate of the functions in the domain of the Laplacian on the Sierpinski gasket has been established in the same paper. The exact inequality for the

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Sierpinski gasket can also be found in [14]. A slight weaker statement for the haromic functions on the domain of the Dirichlet form was shown in general by Teplyaev in [15].

In this paper we first show Hypothesis 8.1 in [12] doesn't hold for the Vicsek set. Secondly we establish sharp Hölder estimates for harmonic functions on a class of connected post critically finite (p.c.f.) self-similar sets, and then show that functions in the domain of Laplacian enjoy the same property.

Let $K$ be the invariant set obtained by an iterated function system $F_i$, $i=1,\cdots,N$, and $V_0=\{p_1,\cdots,p_n\}$ is boundary of $K$ with $n\le N$. If $u$ is a continuous function on $K$, then the initial energy of $K$ is defined by $\varepsilon_0(u,u)=\sum_{1\le i<j\le n}(u(p_i)-u(p_j))^2$. The harnonic extension matrices $D_j$, $j=1,2,\cdots,N$ introduced by Kigami [7–9] satisfy $h|_{F_iV_0}=D_jh|_{V_0}$ if $h$ is harmonic on $K$, and $r=(r_1,r_2,\cdots,r_N)$ is the scaling factor on $K$ (see [9] and [14] for details). The main conclusions on Hölder estimate of harmonic functions in [12] are as follows.

**Theorem 1.1.** *Assume that*

$$\varepsilon_0(D_j\, u,D_j\, u)\le r_j^2\cdot\varepsilon_0(u,u)\quad for\ \ 1\le j\le N. \tag{1.1}$$

*Then*

$$|h(x)-h(y)|\le c\cdot r_w\quad \text{ if }\ x,y\in F_wK, \tag{1.2}$$

*for any harmonic function $h$ and any word $w$, where the constant $c$ may be taken to be a multiple of $\|h\|_\infty$, and and $r_w=r_{w_1}\cdots r_{w_m}$ for $w=w_1\cdots w_m$.*

The inequality (1.1) is the same as the Hypothesis 8.1 in [12] or the Hypothesis 7.3 in [13].

**Theorem 1.2.** *Let $\mu$ be a self-similar measure. If* (1.2) *holds for every harmonic function $h$, then it holds for every function on $dom(\Delta_\mu)$.*

In fact, the inequality in (1.1) is true for quite a large class of Sierpinski gasket type fractal sets, but is not true for the Vicsek set. In [11] Chunwei Pan has showed that it doesn't hold for harmonic functions on the Vicsek set. A varied version of his results are given in Section 2 to show that the Hypothesis 8.1 (the inequality (1.1)) in [12] doesn't hold for the Vicsek set. It would be nice to be able to obtain the Hölder estimate of harmonic functions on general p.c.f. self-similar sets. We also want to obtain it for functions in the domain of the Laplacian on p.c.f. self-similar sets.

The collection of $n$- gaskets and level $n$ Sierpinski gaskets are a subclass of p.c.f. self-similar sets, denoted by $\Psi$. Obviously $\Psi$ is a proper subset of p.c.f. self-similar sets.

For any stochastic matrix $M=(M_{i,j})$, define

$$\delta(M)=\max_j\max_{i_1,i_2}|M_{i_1,j}-M_{i_2,j}|\in[0,1].$$

In [5] Hajnal called $\delta(M)$ the maximum range of $M$. The notation of $\delta(M)$ can also be found in [6].