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Boundedness for Hardy Type Operators on Herz Spaces with Variable Exponents

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Abstract. In this paper, we will prove the boundedness of Hardy type operators $H_{\beta(x)}$ and $H^*_{\beta(x)}$ of variable order $\beta(x)$ on Herz spaces $K^{\alpha(\cdot)}_{p(\cdot),q}$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$, where $\alpha(\cdot)$ and $p(\cdot)$ are both variable.

Key Words: Herz spaces, Hardy type operators, variable exponent.

AMS Subject Classifications: 42B20, 42B35

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1 Introduction

Suppose that $f\in L^1_{loc}(\mathbb{R}^n)$, $0<\beta(x)<n$. The n-dimensional Hardy operator is defined by

$$Hf(x)=\frac{1}{|x|^n}\int_{|t|<|x|}f(t)dt$$

and the Hardy type operators of variable order $\beta(x)$ are defined by

$$\begin{aligned} H_{\beta(x)}f(x)&=\frac{1}{|x|^{n-\beta(x)}}\int_{|t|<|x|}f(t)dt, && x\in\mathbb{R}^n\setminus\{0\},\\ H^*_{\beta(x)}f(x)&=|x|^{\beta(x)}\int_{|t|\geq|x|}\frac{f(t)}{|t|^n}dt, && x\in\mathbb{R}^n\setminus\{0\}, \end{aligned}$$

when $\beta(x)=\beta$, $H_{\beta(x)}=H_\beta$ and $\beta(x)=0$, $H_{\beta(x)}=H$.

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Christ and Grafakos [1] considered the boundedness of H on $L^p(\mathbb{R}^n)$ and obtained the best constant. The paper [2] by Fu, Liu and Lu introduced the n-dimensional fractional Hardy operators and proved the boundedness of commutators on $L^p(\mathbb{R}^n)$ and Herz spaces of homogeneous type. Hardy inequality in the generalized Lebesgue spaces was studied by Samko in [3]. Recently, Zhao, Fu and Lu [4] got endpoint estimates for n-dimensional Hardy operators and commutators.

It is well known that function spaces with variable exponents were intensively studied during the past 20 years, due to their applications to PDE with non-standard growth conditions and so on, we mention e.g., (see [5,6]). A great deal of work has been done to extend the theory of Hardy type operators on the classical Lebesgue spaces to the variable exponent case, (see [7–11]). Lukkassen, Persson, Samko and Wall [12] studied weighted Hardy type inequalities in variable exponent Morrey-type spaces. The boundedness of Hardy type operators on product of Herz-Morrey spaces with variable exponent were investigated by Zhang and Wu in [13]. Izuki (see [14,15]) first introduced the Herz spaces $K^{\alpha}_{p(\cdot),q}$ and $\dot{K}^{\alpha}_{p(\cdot),q}$ with variable exponent p but fixed $\alpha\in\mathbb{R}^n$. Recently, Almeida and Drihem [16] obtained the boundedness of a wide class of sublinear operators, which includes maximal, potential and Calderón-Zygumnd operators on Herz spaces $K^{\alpha(\cdot)}_{p(\cdot),q}$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$ where α and p are both variable. In this paper, we will study the boundedness of the n-dimensional fractional Hardy operators of variable order $\beta(x)$ on Herz spaces $K^{\alpha(\cdot)}_{p(\cdot),q}$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$.

For brevity, C always means a positive constant independent of the main parameters and may change from one occurrence to another. $B(x,r)=\{y\in\mathbb{R}^n:|x-y|<r\}$, $B_k=\{x\in\mathbb{R}^n:|x|<2^k\}$, $R_k=B_k\setminus B_{k-1}$ and $\chi_{R_k}=\chi_k$ be the characteristic function of the set R_k for $k\in\mathbb{Z}$. $|S|$ denotes the Lebesgue measure of S. The exponent $p'(x)$ means the conjugate of $p(x)$, that is, $1/p'(x)+1/p(x)=1$. Let $p^*(x)$ be the Sobolev exponent defined by $1/p^*(x):=1/p(x)-\beta(x)/n$. We write $f\sim g$ if there exist positive constants C such that $C^{-1}g\le f\le Cg$. By $l^q,q\in(0,\infty]$, we denote the discrete Lebesgue space equipped with the usual quasinorm.

To meet the requirements in the next sections, here, the basic elements of the theory of Lebsegue spaces with variable exponent are briefly presented.

Let $p(\cdot):\Omega\to[1,\infty)$ be a measurable function. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega):=\Big\{f \text{ is measurable}:\int_\Omega\Big|\frac{f(x)}{\lambda}\Big|^{p(x)}dx<\infty \text{ for some constant } \lambda>0\Big\}.$$

The space $L^{p(\cdot)}_{loc}(\Omega)$ is defined by

$$L^{p(\cdot)}_{loc}(\Omega):=\Big\{f \text{ is measurable}: f\in L^{p(\cdot)}(K) \text{ for all compact subsets } K\subset\Omega\Big\}.$$