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On a Class of Generalized Sampling Functions

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Abstract. In this note, we discuss a class of so-called *generalized sampling functions*. These functions are defined to be the inverse Fourier transform of a family of piecewise constant functions that are either square integrable or Lebegue integrable on the real number line. They are in fact the generalization of the classic *sinc* function. Two approaches of constructing the generalized sampling functions are reviewed. Their properties such as cardinality, orthogonality, and decaying properties are discussed. The interactions of those functions and Hilbert transformer are also discussed.

Key Words: Generalized sampling function, sinc function, non-bandlimited signal, sampling theorem, Hilbert transform.

AMS Subject Classifications: 41, 42

1 Introduction

In signal processing, the classic *sinc* function is fundamentally significant due to the Shannon sampling theorem [1,9,10]. Recall that the classic sinc function is defined at a number t in the set $\mathbb{R}$ of real numbers by the equation

$$\mathrm{sinc}(t) := \frac{\sin t}{t}.$$

The Shannon sampling theorem enables to reconstruct a *bandlimited signal* from translates of sinc functions weighted by the uniformly spaced samples of that signal. It is natural to ask whether similar sampling theorem exists for *non-bandlimited signals*. To that end, recently many efforts have been made to extend the classic sinc to *generalized sampling functions*, for example, in [3–5]. One kind of generalized sampling functions given in [3], denoted by sinc_H, is defined as the *inverse Fourier transform* of a so-called *symmetric cascade filter*, denoted by H. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}$ be the set

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of integers, and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. Let Z be a subset of $\mathbb{Z}$, a sequence $\boldsymbol{y} := (y_k : k \in Z)$ is said to be in $l^q(Z)$ if and only if

$$\|\boldsymbol{y}\|_{q,Z} := \Big(\sum_{k\in Z} |y_k|^q \Big)^{1/q} < \infty.$$

The symmetric cascade filter H is a piecewise constant function whose value at $\xi \in \mathbb{R}$ is given by

$$H(\xi) := \sum_{n\in\mathbb{Z}_+} b_n \chi_{I_n}(\xi), \tag{1.1}$$

where the sequence $\boldsymbol{b} = (b_n : n \in \mathbb{Z}_+)$ is in $l^2(\mathbb{Z}_+)$, χ_I is the *indicator function* of the set I, and the interval I_n, $n \in \mathbb{Z}_+$, is the union of two symmetric intervals given by the equation

$$I_n := (-(n+1), -n] \cup [n, (n+1)).$$

Let X be a subset of $\mathbb{R}$, and for $q \in \mathbb{N}$, we say a function f is in $L^q(X)$ if and only if

$$\|f\|_{q,X} := \Big(\int_X |f(t)|^q \mathrm{d}t \Big)^{1/q} < \infty.$$

Thus the generalized sampling function sinc_H is defined by the equation

$$\mathrm{sinc}_H := \sqrt{\frac{\pi}{2}} \mathcal{F}^{-1} H, \tag{1.2}$$

where for any signal $f \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$ the Fourier transform of f is given by

$$(\mathcal{F}f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) \mathrm{e}^{-\mathrm{i}\xi t} \mathrm{d}t.$$

Of course, we have that $H \in L^2(\mathbb{R})$ because $\boldsymbol{b} = \in l^2(\mathbb{Z}_+)$. The function $H \in L^2(\mathbb{R})$ implies that $\mathrm{sinc}_H \in L^2(\mathbb{R})$ since the Fourier operator is closed in $L^2(\mathbb{R})$.

The primary purpose of this note is to introduce to interested readers the basic concepts, approaches, properties of the generalized sampling functions, and their potential applications. For the remainder of the note, in Section 2, we review two approaches that lead to generating the generalized sampling functions. In Section 3, we discuss the properties such as cardinality, orthogonality, decaying property of the generalized sampling functions. In Section 4, a sampling formula is discussed concerning non-bandlimited functions in the shift-invariant space of the generalized sampling functions. In Section 5, we explore the interaction of the generalized sampling functions and the Hilbert transform operator.