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Explicit Multi-Symplectic Splitting Methods for the Nonlinear Dirac Equation

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Abstract. In this paper, we propose two new explicit multi-symplectic splitting methods for the nonlinear Dirac (NLD) equation. Based on its multi-symplectic formulation, the NLD equation is split into one linear multi-symplectic system and one nonlinear infinite Hamiltonian system. Then multi-symplectic Fourier pseudospectral method and multi-symplectic Preissmann scheme are employed to discretize the linear subproblem, respectively. And the nonlinear subsystem is solved by a symplectic scheme. Finally, a composition method is applied to obtain the final schemes for the NLD equation. We find that the two proposed schemes preserve the total symplecticity and can be solved explicitly. Numerical experiments are presented to show the effectiveness of the proposed methods.

AMS subject classifications: 65M06, 65M70

Key words: Nonlinear Dirac equation, multi-symplectic method, splitting method, explicit method.

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1 Introduction

In this paper, we consider the $(1+1)$-dimensional nonlinear Dirac (NLD) equation [1]

$$\begin{cases} \Psi_t = A\Psi_x + \mathrm{i}f(|\Psi_1|^2 - |\Psi_2|^2)B\Psi, \\ \Psi_1(x,0) = \phi_1(x), \quad \Psi_2(x,0) = \phi_2(x), \end{cases} \tag{1.1}$$

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where $\Psi=[\Psi_1,\Psi_2]^T$ is a spinorial wave function, which describes a particle with the spin $-1/2$. Here, Ψ_1 and Ψ_2 are complex functions, $\mathrm{i}=\sqrt{-1}$ is the imaginary unit, $f(s)$ is a real function of a real variable s, A and B are matrices

$$A=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Some numerical methods have been developed to solve the NLD equation (1.1), such as spectral methods [2] and finite difference methods [3–5]. In [6], finite volume methods with fine meshes are proposed to study the interaction dynamics of the Dirac solitary waves. In [7–9], high-order accurate Runge-Kutta discontinuous Galerkin method is also developed to simulate the solitary wave interaction of the NLD equation. More recently, an integrating-factor method for the NLD equation is proposed in [10]. In this paper, we aim to study efficient multi-symplectic methods for the NLD equation. Multi-symplectic methods are a kind of methods which can preserve the multi-symplectic conservation law of Hamiltonian partial differential equations (PDEs) under appropriate discretizations and perform better than tranditional methods in long time simulation [11], like the well known symplectic methods (see for instance [12–15]). Recently, such kind of methods have been paid a lot of attentions to [16]. Some multi-symplectic methods have been developed for the Hamiltonian PDEs, such as multi-symplectic Preissmann scheme [11, 17], multi-symplectic Runge-Kutta methods [18], multi-symplectic spectral discretizations [19], multi-symplectic Fourier pseudospectral method [20, 21], multi-symplectic wavelet collocation method [22–25], and so on. However, most of the multi-symplectic methods are implicit and not efficient enough in computation. In order to solve these problems, some efforts have also been made. In [26], splitting method is firstly introduced to reduce the difficulty of solving multi-symplectic methods. The effectiveness of multi-symplectic splitting methods is shown numerically in [27–29]. Using symplectic Runge-Kutta-Nyström methods and symplectic Runge-Kutta-type methods, Hong et al. developed explicit multi-symplectic methods for the wave equation [31] and the Klein-Gordon-Schrödinger equation [32], respectively.

In [33], it is shown that the NLD equation can be written into a multi-symplectic form. And based on such a formulation, multi-symplectic Runge-Kutta (MSRK) methods for the NLD equation are theoretically investigated. Furthermore, numerical experiments are presented to show the effectiveness of the MSRK methods for the NLD equation in [1]. However, the MSRK methods for the NLD equation are implicit. It is required to use a fixed-point iteration method to solve nonlinear equations which will cost a lot of efforts. In this paper, we develop two explicit multi-symplectic splitting methods for the NLD equation. Firstly, the NLD equation is split into one linear subproblem and one nonlinear subproblem. And then, the two subproblems are integrated separately. On the one hand, the linear subproblem is written as a multi-symplectic form. Then, multi-symplectic Fourier pseudospectral method and multi-symplectic Preissmann method are used to discretize this linear subproblem. Moreover, it is shown that the two proposed methods for the linear subproblem can be solved explicitly. On the other hand, the non-