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A Fourier Matching Method for Analyzing Resonances in a Sound-Hard Slab with Subwavelength Holes

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Abstract. This paper presents a Fourier matching method to rigorously study resonances in a sound-hard slab with a finite number of narrow cylindrical holes. The cross sections of the holes, of diameters $\mathcal{O}(h)$ for $h \ll 1$, can be arbitrarily shaped. Outside the slab, a sound field can be represented in terms of its normal derivatives on the apertures of the holes. Inside each hole, the field can be represented in terms of a countable Fourier basis due to the zero Neumann boundary condition on the side surface. The countably infinite Fourier coefficients for all the holes constitute the unknowns. Matching the two field representatives leads to a countable-dimensional linear system governing the unknowns. Due to the invertibility of a principal submatrix of the infinite-dimensional coefficient matrix, we reduce the linear system to a finite-dimensional one. Resonances are those when the finite-dimensional linear system becomes singular. We derive asymptotic formulae for the resonances in the subwavelength structure for $h \ll 1$. They reveal that a sound field with its real frequency close to a resonance frequency can be enhanced by a magnitude $\mathcal{O}(h^{-2})$. Numerical experiments are carried out to validate the proposed resonance formulae.

AMS subject classifications: 35B34, 35B40, 35J05, 35P20

Key words: Acoustic scattering problem, resonance frequency, subwavelength structure, Helmholtz equation, field enhancement.

1 Introduction

Subwavelength structures have attracted great attentions in the area of wave scattering problems in the past decades [2, 9, 10, 12, 15, 16, 29, 32, 35–37]. People have experimentally observed and also numerically verified that subwavelength structures could own some exclusive features, such as extraordinary optical transmission and local field enhancement, showing great potentials in areas such as biological sensing and imaging,

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microscopy, spectroscopy and communication [23, 31]. It has now been well-known that these features are mostly caused by the existence of resonances of high quality factors in subwavelength structures.

Let a scattering problem be defined in a given subwavelength structure. Mathematically, a resonance refers to a complex frequency k, at which the scattering problem loses uniqueness. The quality factor of this resonance is defined by

$$Q=\frac{\mathrm{Re}(k)}{2\mathrm{Im}(k)},$$

which measures how great a wave field of frequency $\mathrm{Re}(k)$ can be enhanced in the structure. Consequently, it is highly desired to design a subwavelength structure with resonances close enough to the real axis.

To this purpose, the existing literature has made great efforts in the past to quantitatively analyze resonances in subwavelength structures by either developing effective computational methods or proposing rigorous mathematical theories [3, 6–8, 11, 14, 17–21, 25–28, 33, 34]. Among the existing theories, roughly two types of methods have been proposed: boundary-integral-equation (BIE) methods and matched-asymptotics (MA) methods, mainly for two-dimensional (2D) subwavelength structures. Bonnetier and Triki [7] developed a BIE method to study resonances in a perfectly conducting half plane with a subwavelength cavity and firstly obtained asymptotic formulae for the resonances. Subsequently, Babadjian *et al.* [3] used this method to study resonances by two interacting subwavelength cavities, Lin and Zhang developed a simplified BIE method to study resonances in a slab with a single slit [26], periodic slits [27, 28], or a periodic array of two subwavelength slits [25], Gao *et al.* [14] studied resonances by a rectangular cavity of mixed conducting parts. Using the MA methods, Joly and Tordeux [19–21] and Clausel *et al.* [11] studied resonances by thin slots, Holley and Schnitzer [17] studied resonances in a slab with a single slit, and Brandão *et al.* [8] studied resonances in a slab of finite conductivity with a single slit or periodic slits. Compared with 2D structures, three-dimensional (3D) subwavelength structures are more flexible in practical fabrication and can in fact realize resonators of higher quality factors [9,15,16,29]. Nevertheless, much fewer theories have been developed so far to rigorously study resonances in 3D structures. Resonances of acoustic waves in a three-dimensional slab of tiny circular or square holes have been studied in [13, 22].

In [38], we proposed a Fourier matching method (FMM) to study resonances in a slab of subwavelength slits. Unlike the existing methods on this topic, the FMM does not use Green's function of each slit, which is complicated. Instead, it takes advantage of the existence of a countable Fourier basis for each slit so that the scattering problem can be reformulated as a countable-dimensional linear system with a frequency-dependent coefficient matrix. Resonance are determined by studying when the coefficient matrix is non-invertible. However, the method relies on explicit asymptotics and delicate estimates of all elements of the matrix, largely limiting its extension to more general structures. To tackle this issue, this paper, using theories of functional analysis as our new tool, gen-