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A Data-Driven Scale-Invariant Weighted Compact Nonlinear Scheme for Hyperbolic Conservation Laws

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Abstract. With continuous developments in various techniques, machine learning is becoming increasingly viable and promising in the field of fluid mechanics. In this article, we present a machine learning approach for enhancing the resolution and robustness of the weighted compact nonlinear scheme (WCNS). We employ a neural network as a weighting function in the WCNS scheme and follow a data-driven approach to train this neural network. Neural networks can learn a new smoothness measure and calculate a weight function inherently. To facilitate the machine learning task and train with fewer data, we integrate the prior knowledge into the learning process, such as a Galilean invariant input layer and CNS polynomials. The normalization in the Delta layer (the so-called Delta layer is used to calculate input features) ensures that the WCNS3-NN schemes achieve a scale-invariant property (Si-property) with an arbitrary scale of a function, and an essentially non-oscillatory approximation of a discontinuous function (ENO-property). The Si-property and ENO-property of the data-driven WCNS schemes are validated numerically. Several one- and two-dimensional benchmark examples, including strong shocks and shock-density wave interactions, are presented to demonstrate the advantages of the proposed method.

AMS subject classifications: 68T07, 35L65, 65M06

Key words: WCNS, neural network, machine learning, scale-invariant, shock-capturing.

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1 Introduction

In the last few decades, computational fluid dynamics (CFD) has attracted much attention in simulating various flow structures with high-order accuracy and high resolution, such as vortex, shockwave, transition, separation, and reattachment. The solution of hyperbolic partial differential equations requires that numerical schemes stably capture discontinuities and resolve small-scale structures with high-fidelity. These two aspects put forward almost contradictory requirements, which renders the development of numerical schemes a long-term challenge.

Weighted essentially non-oscillatory (WENO) scheme is regarded as a suitable choice to address the aforementioned issues. It was first proposed by Liu et al. [1] based on the extension of the ENO scheme [2, 3]. Later, Jiang and Shu [4] designed nonlinear weights and local smoothness to modify the classical finite volume WENO scheme [1]. They developed a fifth-order WENO scheme for hyperbolic conservation laws, which is denoted as WENO-JS. The framework of WENO-JS has been extensively applied in developing a robust numerical method for compressible turbulence [5]. From then on, different improved versions of WENO schemes have been constructed. One kind of method focuses on adjusting or redesigning the nonlinear weights. For example, Henrick et al. [6] pointed out that WENO-JS suffers from a loss of accuracy near smooth extrema, so they constructed a new weighting function and proposed WENO-M such that the overall scheme recovers the ideal order of convergence. Borges et al. [7] developed a higher-order global smoothness indicator and formulated WENO-Z. This scheme allows for better treatment of discontinuities while further minimizing the numerical dissipation and achieving a higher resolution. In addition, there are also several other typical WENO schemes, such as the Hermite WENO scheme [8,9], the Trigonmrtric WENO [10], the WENO-Z+ scheme [11], the central WENO scheme [12], just to name a few.

Another high-resolution and convenient scheme is the weighted compact nonlinear scheme (WCNS) derived by Deng and Zhang [13] for hyperbolic equations. WCNS is an improvement of the compact nonlinear scheme (CNS) proposed by Deng and Maekawa [14]. It combines WENO reconstruction [1] and cell-centered compact schemes [15]. With the help of the conservative metric method (CMM) [16] and the symmetrical conservative metric method (SCMM) [17], WCNS can ensure the geometric conservation law (GCL). WCNS has been successfully applied to complex grid problems [18] and various flow simulations, including boundary layer transition [19], turbulence, and shock boundary layer interactions [20]. In addition, many efforts have been built on the classical WCNS [21–24]. The procedure of WCNS discretization can be described by the following three steps [13]: (i) interpolate the flow variables at nodes to obtain the left and right values at half-nodes, (ii) evaluate the values of flux at half-nodes, and (iii) compute flux derivatives from half-nodes to nodes by using centered differencing schemes. It has been established that the step of nonlinear interpolation is essential for the resolution and shock-capturing capacity of the WCNS scheme [25, 26]. In light of this, studies and improvements for this process have garnered much interest. Here, we focus on the first step of the WCNS