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Multiple Pole Solutions of the Hirota Equation Under Nonzero Boundary Conditions by Inverse Scattering Method

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Abstract. In this paper, we study multiple pole solutions for the focusing Hirota equation under the nonzero boundary conditions via inverse scattering method. The direct scattering problem is based on the spectral analysis and exhibits the Jost solutions, scattering matrix as well as their analyticity, symmetries and asymptotic behaviors. Compared with previous studies, we define a more complex discrete spectrum. The inverse scattering problem is explored by solving the corresponding matrix Riemann-Hilbert problems. Particularly, we solve the scattering problem by a suitable uniformization variable on the complex z-plane instead of a two-sheeted Riemann surface. Finally, we deduce general formulas of N-double pole and N-triple pole solutions with mixed discrete spectra and show some prominent characteristics of these solutions graphically. Our results should be helpful to further explore and enrich breather wave phenomena arising in nonlinear and complex systems.

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Key words: Hirota equation, inverse scattering method, Riemann-Hilbert problem, multiple pole solution.

1. Introduction

The nonlinear Schrödinger (NLS) equation has been known as a ubiquitous mathematical model among many integrable systems. It plays a significant role in nonlinear optics [4, 23], water waves [3, 13], Bose-Einstein condensates [5], and finance [29, 30]. In order to study more abundant physical problems, such as vortex motion [14], the one-dimensional Heisenberg spin system [34], and nonlinear optics [18], Hirota [11] considered the equation

$$\mathrm{i}q_t + \alpha\left(q_{xx} + 2\sigma|q|^2 q\right) + \mathrm{i}\beta\left(q_{xxx} + 6\sigma|q|^2 q_x\right) = 0, \tag{1.1}$$

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which has been named after him later on. Note that α and β are real constants. If $\sigma = 1$, then (1.1) is the focusing Hirota equation, whereas for $\sigma = -1$ it is the defocusing one.

There are various methods for solving the Hirota equation (1.1), e.g. the N-soliton solutions based on the Darboux transformation and arbitrary-order multiple pole solutions using limit technique [15]. We also the inverse scattering transform (IST) [9] and Riemann-Hilbert problem (RHP) based approach [19] applied to integrable nonlinear equations [10, 12, 16, 20, 21, 25–28, 31, 32, 36]. Single and double pole solutions for the focusing Hirota equation with zero boundary condition (ZBC) are discussed in [38] by solving a RHP. Yan *et al.* [35] obtained simple and double pole solutions of the focusing Hirota equation and simple pole solutions of the defocusing one. Zhang and Ling [37] studied the long-time asymptotics of high-order solitons of (1.1).

However, to the best of the authors' knowledge, general multiple pole solitons for the Eq. (1.1) with nonzero boundary conditions (NZBCs) in the case of mixed discrete spectra have not been yet considered. In order to obtain more abundant interactions between solitons and breathers, we investigate the Eq. (1.1) for $\sigma = 1$ and NZBCs

$$\lim_{x\to\pm\infty} q(x,t) = q_{\pm} e^{i[q_0 x + (\alpha q_0^2 - 5\beta q_0^3)t]}, \tag{1.2}$$

where $q_{\pm}$ are constants and $|q_{\pm}| = q_0 > 0$.

Note that the corresponding RHP is non-regular since the reflection coefficient has multiple poles. Nevertheless, such problems can be treated by using a dressing method based on a regularization technique [24, 32, 33]. Besides, the corresponding RHP can be regularized by subtracting pole contribution and asymptotic behavior — cf. [1–3, 6–8, 22]. Therefore, it is natural to employ such an idea for investigating multiple pole solutions.

The rest of this paper is arranged as follows. In Section 2, we discuss direct scattering problems and introduce Jost solutions, a scattering matrix, and recall their properties. In Section 3, we construct a suitable RHP and derive general formulas of N-double pole and N-triple pole solutions. Choosing suitable parameters, we demonstrate remarkable characteristics of one-double and one-triple pole solutions. Conclusions and discussions are carried out in the final section.

2. Direct Scattering Problem

2.1. Lax pair and Jost solutions

The Eq. (1.1) is completely integrable and its Lax pair has the form

$$\begin{aligned} \Phi_x(x,t,\lambda) &= X(x,t,\lambda)\Phi(x,t,\lambda), \\ \Phi_t(x,t,\lambda) &= T(x,t,\lambda)\Phi(x,t,\lambda) \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} X(x,t,\lambda) &= -i\lambda\sigma_3 + Q, \\ T(x,t,\lambda) &= -4i\beta\lambda^3\sigma_3 + \lambda^2(4\beta Q - 2i\alpha\sigma_3) + \lambda\left[2\alpha Q - 2i\beta(Q^2 + Q_x)\sigma_3\right] \\ &\quad + \left[\beta(Q_x Q - Q Q_x) - i\alpha(Q^2 + Q_x)\right]\sigma_3 - \beta Q_{xx} + Q^3. \end{aligned}$$