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Oversampled Collocation Approximation Method of Functions via Jacobi Frames

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Abstract. In this paper, we study the Jacobi frame approximation with equispaced samples and derive an error estimation. We observe numerically that the approximation accuracy gradually decreases as the extended domain parameter γ increases in the uniform norm, especially for differentiable functions. In addition, we show that when the indexes of Jacobi polynomials α and β are larger (for example $\max\{\alpha,\beta\}>10$), it leads to a divergence behavior on the frame approximation error decay.

AMS subject classifications: 41A10, 41A25, 41A17

Key words: Jacobi polynomial, frame, oversampled, collocation, equispaced sample.

1 Introduction

It is well-known that polynomial interpolation of functions at $m+1$ equispaced nodes will lead to the Runge's phenomenon, and many methods have been proposed to overcome the Runge's phenomenon (see, e.g., [4, 5, 7] and references therein). However, all these methods can not circumvent the conclusion of the impossibility theorem which was proved in [16], that is, any approximation procedure that achieves exponential convergence at a geometric rate must also be exponentially ill-conditioned at a geometric rate. Furthermore, as shown in [16], the best possible rate of convergence of a stable method is root-exponential in m.

Recently, Adcock, Huybrechs and Shadrin proposed an approach termed polynomial frame approximation [2, 6]. For some fixed $\gamma>1$, polynomial frame approximation uses orthogonal polynomials on an extended interval $[-\gamma,\gamma]$ to construct an approximation to a function over $[-1,1]$. This method leads to an ill-conditioned least-squares problem, while Adcock and Shadrin showed that this problem can be computed accurately via a

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regularized singular value decomposition (SVD) at a set of $m+1$ linear oversampling equispaced nodes on $[-1,1]$, and proved that the regularized frame approximation operator is well-conditioned [6]. Further, the two authors also showed that the exponential decay of the polynomial frame approximation error down to a finite user-determined tolerance $\varepsilon>0$ is indeed possible for functions that are analytic in a sufficiently large region. In other words, Adcock and Shadrin asserted the possibility of fast and stable approximation of analytic functions from equispaced samples [6].

When studying the theoretical analysis of polynomial frame approximation, Adcock and Shadrin adopted the Legendre polynomials for convenience [6]. To explore the generality of the polynomial frame approximation, we next consider the use of Jacobi polynomials as well as their special cases, including Chebyshev, Legendre and Gegenbauer polynomials, which are widely used in many branches of scientific computing such as approximation theory, Gauss-type quadrature and spectral methods for differential and integral equations (see, e.g., [8, 14, 17, 18]). Among these applications, Jacobi polynomials are particularly appealing owing to their superior properties: (i) they have excellent error properties in the approximation of a globally smooth function; (ii) quadrature rules based on their zeros or extrema are optimal in the sense of maximizing the exactness of polynomials.

In this paper, we derive the Jacobi frame approximation error bound in Section 3 and we focus on the numerical experiments with various extended domain parameter γ in Section 4, in particular for differentiable functions. The Jacobi frame approximation accuracy will be gradually lost as γ increases, and it can be found that the higher the smoothness of the approximated function, the more obvious the loss of approximation accuracy. Further, we also observe numerically that when the parameters of Jacobi polynomials α, β are larger, for example $\mu=\max\{\alpha,\beta\}>10$, the approximation error will become worse or even divergent.

The paper is organized as follows. In Section 2 we state the Jacobi frame approximation, and we derive the approximation results in Section 3. We then present a large number of numerical experiments of analytic functions and differentiable functions in Section 4. The final section contain concluding remarks.

2 Preliminaries

2.1 Notation

Let $\mathbb{P}_n$ denotes the space of polynomials of degree at most n and $C(I)$ denotes the space of continuous functions on interval $I=[-1,1]$. The uniform norm over I is defined by

$$\|g\|_{I,\infty}=\sup_{x\in I}|g(x)|, \quad g\in C(I).$$

For weight function

$$w^{(\alpha,\beta)}(x)=(1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\beta>-1, \tag{2.1}$$