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A Numerical Study of Integrated Linear Reconstruction for Steady Euler Equations Based on Finite Volume Scheme

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Abstract. Towards the solution reconstruction, one of the main steps in Godunov type finite volume scheme, a class of integrated linear reconstruction (ILR) methods has been developed recently, from which the advantages such as parameters free and maximum principle preserving can be observed. It is noted that only time-dependent problems are considered in the previous study on ILR, while the steady state problems play an important role in applications such as optimal design of vehicle shape. In this paper, focusing on the steady Euler equations, we will extend the study of ILR to the steady state problems. The numerical framework to solve the steady Euler equations consists of a Newton iteration for the linearization, and a geometric multigrid solver for the derived linear system. It is found that even for a shock free problem, the convergence of residual towards the machine precision can not be obtained by directly using the ILR. With the lack of the differentiability of reconstructed solution as a partial explanation, a simple Laplacian smoothing procedure is introduced in the method as a post-processing technique, which dramatically improves the convergence to steady state. To prevent the numerical oscillations around the discontinuity, an efficient WENO reconstruction based on secondary reconstruction is employed. It is shown that the extra two operations for ILR are very efficient. Several numerical examples are presented to show the effectiveness of the proposed scheme for the steady state problems.

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1 Introduction

Steady Euler equations play an important role in aerodynamic shape optimization problems [1,14], and have been viewed as a benchmark problem to evaluate the performance of various numerical schemes in computational fluid dynamics (CFD), see [4,24,37,44,45] among others. A lot of numerical methods have been developed to solve the unsteady and steady Euler equations, e.g., the discontinuous Galerkin method [4,33,50,51], the finite volume method [3,22,36,44], the spectral volume method [47], and the fast sweeping method [11,12].

Nowadays, the second-order finite volume scheme is one of the most popular schemes to solve conservation laws. The second-order finite volume method is a generalization of the classical first-order Godunov's method [24] and mainly includes three steps, i.e., first a piecewise linear polynomial is reconstructed in each cell by using cell averages in the reconstruction patch, then the governing equation is evolved, and finally the cell average is updated in each cell. When the solutions of conservation laws contain discontinuities, how to effectively reduce or prevent spurious oscillations around the discontinuities needs special attention in the reconstruction step. To date, several pioneering works are available in the literature to prevent the numerical oscillations. For example, the total variation diminishing (TVD) [16] limiters have been successfully used for one-dimensional problems to prevent spurious oscillations and to achieve second-order accuracy. However, the implementation of TVD limiter for multi-dimensional problems on unstructured meshes is non-trivial. Moreover, the TVD schemes have been proved to be at most first-order accurate for 2D scalar conservation laws [15]. To overcome these drawbacks, the slope limiters have been introduced to make the numerical solution monotone. The process of slope limiting usually consists of two components, i.e., given the cell averages in the related reconstruction patch of a target cell, the unlimited gradient in this cell is determined by Green-Gauss method [3] or least-squares reconstruction (also known as k-exact reconstruction) [2], and the unlimited gradient is then limited by a certain process to prevent numerical oscillations. On structured meshes, limiters such as the minmod limiter, the Superbee limiter and the van Leer limiter are routinely used to prevent numerical oscillations, and we refer to [24] for the details. On the other hand, the pioneering work of slope limiter on unstructured meshes was due to Barth and Jespersen [3]. It is well known that the non-differentiability of Barth and Jespersen limiter hampers the convergence to steady state [44]. To resolve this issue, the Venkatakrishnan limiter [44] was proposed. However, since the limiter does not preserve strict monotonicity [19,44], slight oscillations can be observed near strong shocks when using this limiter. In order to obtain high-order numerical accuracy and to effectively prevent spurious oscillations, the essentially non-oscillatory (ENO) and the weighted ENO