East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2023-075.010623

Vol. **14**, No. 1, pp. 147-178
February 2024

Numerical Shape Reconstruction for a Semi-Linear Elliptic Interface Inverse Problem

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Received 27 February 2023; Accepted (in revised version) 1 June 2023.

Abstract. We consider a shape reconstruction inverse problem constrained by a semi-linear elliptic interface problem in reaction diffusion. The existence of the model is shown. We perform shape sensitivity analysis and propose two numerical optimization algorithms based on the distributed shape gradient. The first algorithm allows shape changes and the second algorithm uses a level set method allowing shape and topological changes. Numerical results are presented to verify effectiveness of the algorithms.

AMS subject classifications: 49M41, 49N45

Key words: Shape reconstruction, distributed shape gradient, interface problem, semi-linear elliptic, level set method.

1. Introduction

Theoretical and numerical aspects of geometric inverse problems with unknown geometric shape have been investigated for many years. In various applications, an unknown subdomain and its complement are the sets where discontinuous parameters takes different constant values. Geometric inverse problems have wide applications in engineering, including heat source identification [14, 24], interface reconstruction of diffusive coefficient [16, 19, 21, 32], elastic inclusion detection [2], and electrical impedance tomography [10, 20].

For solving such problems, one needs to find a mechanism to represent the shape and follow its evolution. A straightforward method for solving such problems consists in parametrization of the shape, usually curve/surface and follow its evolution after regularization. However, this simple approach may fail if no topology information on recon-

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struction is known a priori, especially for reconstruction of possible multi-connected components. Then, approaches and techniques allowing shape and topological changes are required, such as the level set method [1,8], phase-field method [5], and topological derivative approach [18,34]. A level set method was developed for numerically solving elliptic interface inverse problem [8,10,19] and inverse obstacle problem [30].

Shape optimization and reconstruction has caused interests not only on linear partial differential equation (PDE) constraints, but also on nonlinear PDE cases [12,18,31,34,40]. Interface location reconstruction was studied associated with linear elliptic problems [10, 19] and linear elasticity [2,35]. For interface identification of semi-linear elliptic problems, sensitivity analysis was explored with the topological derivative [3] (see [5] for phase-field approach). Shape optimization of semi-linear elliptic problems is considered in [18,31,40]. In this paper, we consider an interface inverse problem associated with a semi-linear elliptic interface problem for chemical or heat reaction diffusion. We show existence of a solution to the optimization model and propose two numerical algorithms for reconstruction when the subdomain components are simply connected or multi-connected. Both the boundary moving and the level set method we propose rely on the objective's distributed shape gradient. The so-called shape gradient in the shape optimization community is contained in the Eulerian derivative, which measures the sensitivity information of the objective with respect to domain (shape) variations [33]. The Eulerian derivative is known to have two forms — viz. boundary type and volumetric one. The latter holds more generally, although they are equivalent when the boundary has certain smoothness [6,11]. Their discrete finite-element approximations behave different even when their formulations are equivalent on the continuous level. The boundary expression may work not well in certain optimization algorithms [38] (see [15,39] for possible theoretical explanations on the accuracy advantage of finite-element discretizations of distributed shape gradients). The distributed shape gradient has been applied in many problems including electrical impedance tomography [4,22], structural topology optimization [7], shape design of fluid flows [23], etc.

The rest of the paper is organized as follows. In Section 2, we introduce the interface inverse problem associated with a semi-linear elliptic boundary value problem. Then we show existence and perform shape sensitivity analysis. In Section 3, we propose two numerical algorithms based on shape gradient and level set method. In Section 4, numerical examples are presented to test performance of both algorithms. Brief conclusions follow in last section.

2. Existence and Shape Sensitivity of Model Problem

We first introduce an interface reconstruction inverse problem in nonlinear reaction diffusion, which can have applications in chemical diffusion processes or heat conduction. Then we show existence and perform shape sensitivity analysis with shape calculus. Let us first introduce notations for Sobolev spaces. Let $D \subset \mathbb{R}^d$, $d = 2,3$ be an open bounded domain with Lipschitz boundary ∂D. The $L^\nu(D)$ function spaces are defined by

$$L^\nu(D) := \left\{ u : \|u\|_{L^\nu(D)} < \infty \right\}, \quad \nu = 2, \infty,$$