East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2022-353.260523

Vol. **14**, No. 1, pp. 124-146
February 2024

# Analysis of a System of Hemivariational Inequalities Arising in Non-Stationary Stokes Equation with Thermal Effects

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*Received 22 December 2022; Accepted (in revised version) 26 May 2023.*

**Abstract.** A non-stationary Stokes equation coupled with an evolution equation of temperature field is studied. Boundary conditions for velocity and temperature fields contain the generalized Clarke gradient. The corresponding variational formulation is governed by a system of hemivariational inequalities. The existence and uniqueness of a weak solution is proved by employing Banach fixed point theorem and hemivariational inequalities. Besides, a fully-discrete problem for this system of hemivariational inequalities is given and error estimates are derived.

**AMS subject classifications**: 65M15

**Key words**: Non-stationary Stokes equation, hemivariational inequality, thermal effects, Banach fixed point theorem, numerical analysis.

## 1. Introduction

Hemivariational inequalities, as a generation of variational inequalities, constitute important tools in studying various nonlinear problems arising in chemistry, physics, biology, engineering, and many other fields. Research on variational inequalities stems from the monotonicity theory and convexity theory, while the study of hemivariational inequalities employs the Clarke subdifferential property of locally Lipschitz functions as the main component and allows the inclusion of non-convex functions. On the one hand, hemivariational inequalities have more advantages than variational inequalities in the characterization of some practical problems, and on the other hand, benefit from the development of non-smooth analysis and multivalued analysis, the theoretical and numerical analysis of hemivariational inequality develop rapidly in past few decades. In particular, variational or

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hemivariational inequalities arising in contact mechanics attracted widespread interests — cf. Refs. [4,6,13,22,24]. On the other hand, many researchers are interested in applying hemivariational inequalities to fluid mechanic problems [8,9,17,28]. The paper [8] studies a hemivariational inequality arising from a stationary Stokes equation equipped with a nonlinear slip boundary condition, the finite element method is employed to solve the hemivariational inequality and error estimates are provided. Paper [9] is devoted to studying a type of hemivariational inequalities that arises in a non-stationary Navier-Stokes problem. Existence of solution for the abstract hemivariational inequality is sought by a kind of time discretization method, termed as Rothe method. Paper [17] studies Stokes problem for a generalized Newtonian fluid along with unilateral, slip and leak boundary conditions. Existence of a unique weak solution is proven through a surjectivity theorem. A Stokes problem for an incompressible fluid, whose boundary conditions are in the type of subdifferential was stadied in [28]. The associated variational formulation forms a variational-hemivariational inequalities system. The corresponding solution existence as well as the weak compactness of the solution set is established by Schauder fixed point theorem.

The above papers do not consider the interaction of velocity and temperature fields of the fluid, so that the only a pure fluid dynamics problem is studied. However, in fact many parameters of actual fluids are affected by the temperature. In contrast, the flow of fluids also causes changes in temperature. Therefore, numerous studies focus on the fluid problems with thermal effects — cf. [2,20,21,27]. In these papers, either Dirichlet or Neumann boundary conditions are considered, so that all the models lead to a system of equations. However, the physical phenomena can be multitudinous and various boundary conditions can be required. Thus assuming the boundary conditions to include subdifferential nonconvex functions, we can arrive at a system of hemivariational inequalities. Hitherto, there is no works considering hemivariational inequalities arising from Stokes flow with thermal effects, and our aim is to cover this gap.

More exactly, this paper focuses on variational and numerical analysis of a system of hemivariational inequalities arising in a non-stationary incompressible Stokes equation coupled with an evolution equation of temperature field. Inspired by the ideas of [15,16,21], we consider the following conservation laws:

$$
\begin{aligned}
&\boldsymbol{u}'(t) - \nu^* \Delta \boldsymbol{u}(t) + \nabla p(t) - c_e \theta(t) = \boldsymbol{q}(t) && \text{in } \Theta \times (0,T),\\
&\operatorname{div} \boldsymbol{u}(t) = \boldsymbol{0} && \text{in } \Theta \times (0,T),\\
&\theta'(t) - \Delta\theta(t) = -c_{ij}\frac{\partial u_i}{\partial x_j}(t) + g(t) && \text{in } \Theta \times (0,T),
\end{aligned}
$$

where $\Theta \subset \mathbb{R}^d, d = 2,3$ is a bounded connected domain, whose boundary $\Gamma$ is Lipschitz continuous and $0 < T < \infty$. Besides, $\boldsymbol{u}(\boldsymbol{x},t)$ is the flow velocity, $\boldsymbol{q}(\boldsymbol{x},t)$ an external force, $\nu^*$ a positive viscosity constant, $p(\boldsymbol{x},t)$ the pressure, $\theta(\boldsymbol{x},t)$ the temperature, $g(\boldsymbol{x},t)$ the density of volume heat sources, and $c_e = (c_{ij})$ the thermal influence operator. Subsequently, the boundary conditions are made up as follows:

$$
\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{1.1}
$$

$$
\theta(t) = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{1.2}
$$