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The Neumann Problem for a Class of Fully Nonlinear Elliptic Partial Differential Equations

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Abstract. In this paper, we establish global C^2 estimates to the Neumann problem for a class of fully nonlinear elliptic equations. As an application, we prove the existence and uniqueness of k-admissible solutions to the Neumann problems.

Key Words: Neumann problem, fully nonlinear, elliptic equation.

AMS Subject Classifications: 35J60, 35J09, 35J40

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain and $\nu(x)$ be the outer unit normal at $x \in \partial\Omega$. Suppose $f \in C^2(\Omega)$ is a positive function and $a, b \in C^3(\partial\Omega)$ with $a > 0$. In this paper, we consider the Neumann problem of the fully nonlinear equation

$$\begin{cases} S_k(W) = f(x), & \Omega, \\ u_\nu = -a(x)u + b(x), & \partial\Omega, \end{cases} \tag{1.1}$$

where the matrix $W = (w_{\alpha_1\cdots\alpha_m,\beta_1\cdots\beta_m})_{C_n^m \times C_n^m}$ with the elements as follows, for $1 \leq m \leq n-1$,

$$w_{\alpha_1\cdots\alpha_m,\beta_1\cdots\beta_m} = \sum_{\gamma=1}^{n}\sum_{i=1}^{m} u_{\gamma\alpha_i}\delta^{\alpha_1\cdots\alpha_{i-1}\gamma\alpha_{i+1}\cdots\alpha_m}_{\beta_1\cdots\beta_{i-1}\beta_i\beta_{i+1}\cdots\beta_m}, \tag{1.2}$$

a linear combination of u_{ij}, where $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\delta^{\alpha_1\cdots\alpha_{i-1}\gamma\alpha_{i+1}\cdots\alpha_m}_{\beta_1\cdots\beta_{i-1}\beta_i\beta_{i+1}\cdots\beta_m}$ is the generalized Kronecker symbol. All indexes $\alpha_1, \beta_1, \cdots$ come from 1 to n. For each $1 \leq k \leq C_n^m$, we define

$$S_k(W) = S_k\big(\lambda(W)\big) = \sum_{1\leq i_1<i_2<\cdots<i_k\leq C_n^m} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k},$$

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where $\lambda(W)=(\lambda_1,\lambda_2,\cdots,\lambda_{C_n^m})$ is the set of eigenvalues of W. We also set $S_0(W)=1$.

In fact, the matrix W comes from the following operator $U^{[m]}$ as in [2] and [10]. First, we note that $(u_{ij})_{n\times n}$ induces an operator U on $\mathbb{R}^n$ by

$$U(e_i)=\sum_{j=1}^{n}u_{ij}e_j,\quad \forall 1\leq i\leq n,$$

where $\{e_1,e_2,\cdots,e_n\}$ is the standard basis of $\mathbb{R}^n$. We further extend U to act on the real vector space $\wedge^m\mathbb{R}^n$ by

$$U^{[m]}(e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_m})=\sum_{i=1}^{m}e_{\alpha_1}\wedge\cdots\wedge U(e_{\alpha_i})\wedge\cdots\wedge e_{\alpha_m},$$

where $\{e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_m}|1\leq\alpha_1<\cdots<\alpha_m\leq n\}$ is the standard basis for $\wedge^m\mathbb{R}^n$. Then W is the matrix of $U^{[m]}$ under this standard basis. It is convenient to denote the multi-index by $\bar{\alpha}=(\alpha_1\cdots\alpha_m)$. We only consider the admissible multi-index, that is, $1\leq\alpha_1<\alpha_2\cdots<\alpha_m\leq n$. By the dictionary arrangement, we can arrange all admissible multi-indexes from 1 to C_n^m and use $N_{\bar{\alpha}}$ denote the order number of the multi-index $\bar{\alpha}=(\alpha_1\cdots\alpha_m)$, i.e., $N_{\bar{\alpha}}=1$ for $\bar{\alpha}=(12\cdots m)$, $\cdots$. We also use $\bar{\alpha}$ denote the index set $\{\alpha_1,\cdots,\alpha_m\}$. It is not hard to see that

$$W_{N_{\bar{\alpha}}N_{\bar{\alpha}}}=w_{\bar{\alpha}\bar{\alpha}}=\sum_{i=1}^{m}u_{\alpha_i\alpha_i}, \tag{1.3a}$$

$$W_{N_{\bar{\alpha}}N_{\bar{\beta}}}=w_{\bar{\alpha}\bar{\beta}}=(-1)^{|i-j|}u_{\alpha_i\beta_j}, \tag{1.3b}$$

when the index set $\{\alpha_1,\cdots,\alpha_m\}\setminus\{\alpha_i\}$ equals to the index set $\{\beta_1,\cdots,\beta_m\}\setminus\{\beta_j\}$ but $\alpha_i\neq\beta_j$; and also

$$W_{N_{\bar{\alpha}}N_{\bar{\beta}}}=w_{\bar{\alpha}\bar{\beta}}=0, \tag{1.4}$$

when the index sets $\{\alpha_1,\cdots,\alpha_m\}$ and $\{\beta_1,\cdots,\beta_m\}$ have more than one different element. It follows that W is symmetrical and diagonal with $(u_{ij})_{n\times n}$ diagonal. The eigenvalues of W are the m-sums of eigenvalues of $(u_{ij})_{n\times n}$.

Define the Gårding's cone in $\mathbb{R}^n$ by

$$\Gamma_k=\{\mu\in\mathbb{R}^n|S_i(\mu)>0,\ \forall 1\leq i\leq k\},\quad 1\leq k\leq n.$$

For $\mu\in\mathbb{R}^n$, we let

$$\Psi=\Psi(\mu)=\{\mu_{i_1}+\cdots+\mu_{i_m}|1\leq i_1<\cdots<i_m\leq n\}\in\mathbb{R}^{C_n^m}.$$

Then we define the generalized Gårding's cone as follows,

$$\Gamma_k^{(m)}=\{\mu\in\mathbb{R}^n|S_i(\Psi)>0,\ \forall 1\leq i\leq k\},\quad 1\leq m\leq n,\quad 1\leq k\leq C_n^m. \tag{1.5}$$