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General Optimal Polynomial Approximants, Stabilization, and Projections of Unity

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Abstract. In various Hilbert spaces of analytic functions on the unit disk, we characterize when a function has optimal polynomial approximants given by truncations of a single power series or, equivalently, when the approximants stabilize. We also introduce a generalized notion of optimal approximant and use this to explicitly compute orthogonal projections of 1 onto certain shift invariant subspaces.

Key Words: Optimal polynomial approximants, inner functions.

AMS Subject Classifications: 46E22, 0J05

1 Background, introduction, and notation

Throughout this paper $\mathcal{H}$ will be a reproducing kernel Hilbert space of analytic functions on the unit disk $\mathbb{D}$. We will denote the reproducing kernel for $\mathcal{H}$ as $k_\lambda(z) = k(z,\lambda)$ and the normalized reproducing kernel as $\hat{k}_\lambda = k_\lambda/\|k_\lambda\|_{\mathcal{H}}$. That is, a priori, for $\lambda \in \mathbb{D}$, we have $f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}}$. Further, we will assume that $\mathcal{H}$ satisfies the following:

1. The polynomials $\mathcal{P}$ are dense in $\mathcal{H}$.

2. The forward shift S, mapping $f(z) \mapsto zf(z)$, is a bounded operator on $\mathcal{H}$.

When $V \subseteq \mathcal{H}$ is a closed subspace, we will use $\Pi_V : \mathcal{H} \to V$ to denote the orthogonal projection from $\mathcal{H}$ onto V. For $n \in \mathbb{N}$, we will denote by $\mathcal{P}_n$ the set of complex polynomials of degree less than or equal to n. For $f \in \mathcal{H}$, we define $f\mathcal{P}_n := \{pf : p \in \mathcal{P}_n\}$. Note that $f\mathcal{P}_n$ is always a closed finite-dimensional subspace of $\mathcal{H}$. When f is fixed, we will use $\Pi_n : \mathcal{H} \to f\mathcal{P}_n$ to denote the orthogonal projection onto $f\mathcal{P}_n$.

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1.1 Cyclicity and shift invariant subspaces

The results to come are born from the study of shift invariant subspaces and cyclic functions. We say a subspace $V \subseteq \mathcal{H}$ is shift invariant if $SV \subseteq V$. We say a function $f \in \mathcal{H}$ is cyclic (in $\mathcal{H}$) if

$$[f] := \overline{\text{span}\{z^n f : n \geq 0\}}^{\mathcal{H}}$$

is equal to $\mathcal{H}$ itself. Note that $[f]$ is a (possibly trivial) shift invariant subspace and is the smallest closed subspace of $\mathcal{H}$ containing f. In [7], it was pointed out that $f \in \mathcal{H}$ is cyclic if and only if, for any cyclic function $g \in \mathcal{H}$, there exist polynomials $(p_n)_{n\geq 0}$ so that $\|p_n f - g\|_{\mathcal{H}} \to 0$. From this equivalence, and taking $g = 1$ in spaces where $1 = k_0$, the study of optimal polynomial approximants has arisen. The optimality referred to here is with respect to the distance between $f\mathcal{P}_n$ and 1, i.e.,

$$\min_{p\in\mathcal{P}_n} \|pf - 1\|_{\mathcal{H}}.$$

The element of $f\mathcal{P}_n$ minimizing this distance will be denoted $p_n^* f$ (details to come in Section 2).

Approximation problems of this kind were first studied under the engineering lens of filter design in the 1970's and 80's, referred to as least squares inverses (see, e.g., [8,9,15]). It seems this body of work was not known to mathematicians prior to the discussion in [6].

A modern jumping off point for optimal approximants could be considered the work in [12]; the authors study the optimal approximants of the function $1 - z$ in order to characterize the cyclicity of holomorphic functions on the closed unit disk. In [6], the authors compute Taylor coefficients of $1 - p_n^* f$ in weighted Hardy spaces (discussed below) when f is a polynomial, proving results about the convergence of $(1 - p_n^* f)$.

In [4], the authors study a larger class of reproducing kernel Hilbert spaces and give results on accumulation points, along with lower bounds on the moduli of zeros of optimal approximants. Then in [5], the authors dive into orthogonal polynomials and reproducing kernels in order to get lower bounds on the moduli of zeros of optimal approximants in Dirichlet-type spaces.

Following these themes, we would like to develop some theory for different choices of g (cyclic or not) in considering $\|pf - g\|_{\mathcal{H}}$, and then explore the relationship between optimal approximants and generalized inner functions (this relationship first studied in [3]). This will then yield some observations which allow us to explicitly compute $\Pi_{[f]}(1)$ when f is a polynomial.

In particular: Section 2 develops the framework necessary for handling general optimal approximants. Section 3 deals with stabilization of optimal approximants to $\hat{k}_0/f$, with Theorem 3.1 characterizing when $p_n^* f = p_M^* f$ for all n great than some fixed $M \geq 0$. Section 4 discusses stabilization of general optimal approximants, with Theorem 4.1 giving a version of Theorem 3.1 for general approximants. Section 5 develops the theory of reproducible points, and then returns to certain spaces where $\hat{k}_0 = 1$, with Theorem 5.1