Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **16**, No. 1, pp. 24-46

DOI: 10.4208/aamm.OA-2022-0019
February 2024

A Nitsche-Based Element-Free Galerkin Method for Semilinear Elliptic Problems

Tao Zhang[1] and Xiaolin Li[2,3,*]

[1] *School of Mathematics, Physics and Data Science, Chongqing University of Science and Technology, Chongqing 401331, China*
[2] *School of Mathematical Sciences, Chongqing Normal University, Chongqing 400047, China*
[3] *Key Laboratory for Optimization and Control Ministry of Education, Chongqing Normal University, Chongqing 400047, China*

Received 20 January 2022; Accepted (in revised version) 20 December 2022

Abstract. A Nitsche-based element-free Galerkin (EFG) method for solving semilinear elliptic problems is developed and analyzed in this paper. The existence and uniqueness of the weak solution for semilinear elliptic problems are proved based on a condition that the nonlinear term is an increasing Lipschitz continuous function of the unknown function. A simple iterative scheme is used to deal with the nonlinear integral term. We proved the existence, uniqueness and convergence of the weak solution sequence for continuous level of the simple iterative scheme. A commonly used assumption for approximate space, sometimes called inverse assumption, is proved. Optimal order error estimates in L^2 and H^1 norms are proved for the linear and semilinear elliptic problems. In the actual numerical calculation, the characteristic distance h does not appear explicitly in the parameter β introduced by the Nitsche method. The theoretical results are confirmed numerically.

AMS subject classifications: 65N15, 65N30

Key words: Meshless method, element-free Galerkin method, Nitsche method, semilinear elliptic problem, error estimate.

---

1 Introduction

Numerical methods are requisite and useful for the study of semilinear partial differential equations (PDEs) [1]. The nonlinearity of the semilinear problems only involves the unknown function, not its derivative. Many works have been devoted to the numerical

*Corresponding author.
Emails: zhangtao895701777@126.com (T. Zhang), lxlmath@163.com (X. Li)

solutions of semilinear elliptic problems such as finite element method (FEM) [2, 3], finite difference method [4], finite volume element method [5] and discontinuous Galerkin method [6]. Recently, some collocation meshless (or meshfree) methods [7, 8], Galerkin-type meshless method [8] and generalized finite difference method [9, 10] have been developed to solve the semilinear PDEs. Unlike mesh-based numerical methods, the shape functions used in the meshless methods [11–14] are linkage with nodes (or particles) scattered in the underlying computational domain, which reduces the dependence on the mesh. The meshless methods have greatly developed in the last three decades.

The element-free Galerkin (EFG) [14] method is a global Galerkin-type meshless discretization technique for PDEs. The EFG shape functions are derived from the moving least-squares (MLS) approximation [15]. The difficulty with the imposition of essential (or Dirichlet) boundary conditions stems from the fact that the MLS shape functions are not interpolating. That is, the shape function associated with a node does not vanish at other nodes. Recently, some variants of the MLS approximation have been developed to regain interpolating properties, e.g., interpolating MLS method [15], simplified interpolating MLS method [18, 19], and improved interpolating MLS method [20], and smoothed MLS approximation [21]. On the other hand, the EFG method have been developed for solving solute transport problems [22], tumor growth model [23] and heat transport equation [24], as well as some nonlinear models, such as magnetohydrodynamics(MHD) [25] and Korteweg-de Vries-Rosenau-regularized long-wave equations [26].

In addition to adopting the interpolating shape functions, some mandatory methods, such as the Lagrange multiplier method [12–14], the penalty method [12, 13, 16, 17, 27, 28] and Nitsche method [29–33, 35], can straightforwardly use the non-interpolating shape functions by modifying the original weak form. The Nitsche method was first introduced in early 70's in FEM context [29]. This approach seems to be more promising because of its ease of implementation, its smaller parameter-value compared with the penalty method, its maintenance in terms of the number of unknown variables and the symmetry positive definiteness of the resulting system. Therefore, the Nitsche method is seen as a consistent improvement of the penalty method [31], and these potential advantages bring some conveniences for numerical analysis.

There are a few theoretical results in the Nitsche-based meshless method. In Ref. [32], the approximation errors of the Galerkin meshless method for linear elliptic problem are analyzed based on the nonsymmetric Nitsche method and an inverse assumption, and the effect of the numerical integration are discussed. The error estimates combined with the effect of numerical integration are also developed in [33, 34] based on the reproducing kernel gradient smoothing integration method. Using the Nitsche method, a fast time discrete EFG method is analyzed for the fractional diffusion-wave equation [35]. In these currently reported works, however, the parameter β introduced by the Nitsche method is empirical rather than rational, meanwhile, an unproven inverse assumption is required in the Nitsche-based meshless numerical analysis. Moreover, the analysis presented in [32, 33, 35] addresses only the linear PDEs.

A Nitsche-based element-free Galerkin method is presented in this paper for semi-