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Uniform Error Bound of an Exponential Wave Integrator for Long-Time Dynamics of the Nonlinear Schrödinger Equation with Wave Operator

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Abstract. We establish a uniform error bound of an exponential wave integrator Fourier pseudospectral (EWI-FP) method for the long-time dynamics of the nonlinear Schrödinger equation with wave operator (NLSW), in which the strength of the nonlinearity is characterized by ε^{2p} with $\varepsilon \in (0,1]$ a dimensionless parameter and $p \in \mathbb{N}^+$. When $0 < \varepsilon \ll 1$, the long-time dynamics of the problem is equivalent to that of the NLSW with $\mathscr{O}(1)$-nonlinearity and $\mathscr{O}(\varepsilon)$-initial data. The NLSW is numerically solved by the EWI-FP method which combines an exponential wave integrator for temporal discretization with the Fourier pseudospectral method in space. We rigorously establish the uniform H^1-error bound of the EWI-FP method at $\mathscr{O}(h^{m-1} + \varepsilon^{2p-\beta}\tau^2)$ up to the time at $\mathscr{O}(1/\varepsilon^\beta)$ with $0 \le \beta \le 2p$, the mesh size h, time step τ and $m \ge 2$ an integer depending on the regularity of the exact solution. Finally, numerical results are provided to confirm our error estimates of the EWI-FP method and show that the convergence rate is sharp.

AMS subject classifications: 35Q55, 65M12, 65M15, 65M70, 81-08

Key words: Nonlinear Schrödinger equation with wave operator, long-time dynamics, exponential wave integrator, Fourier pseudospectral method, uniform error bound.

1. Introduction

In this paper, we consider the following nonlinear Schrödinger equation with wave operator on the torus $\mathbb{T}^d$ $(d = 1,2,3)$:

$$
\begin{aligned}
& i\partial_t\psi - \alpha\partial_{tt}\psi + \nabla^2\psi - \varepsilon^{2p}|\psi|^{2p}\psi = 0, \quad \mathbf{x} \in \mathbb{T}^d, \quad t > 0, \\
& \psi(\mathbf{x},0) = \psi_0(\mathbf{x}), \quad \partial_t\psi(\mathbf{x},0) = \psi_1(\mathbf{x}), \quad \mathbf{x} \in \mathbb{T}^d,
\end{aligned}
\tag{1.1}
$$

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where $\psi := \psi(\mathbf{x}, t)$ is a complex-valued wave function with the spatial variable $\mathbf{x}$ and time t, $\alpha = \mathcal{O}(1)$ a positive constant, $0 < \varepsilon \le 1$ a dimensionless parameter controlling the strength of the nonlinearity, $\nabla^2 = \Delta$ the d-dimensional Laplace operator, and $p \in \mathbb{N}^+$. In addition, $\psi_0(\mathbf{x}) = \mathcal{O}(1)$ and $\psi_1(\mathbf{x}) = \mathcal{O}(1)$ are two given complex-valued functions representing the initial wave and velocity, respectively. The solution of the NLSW with weak nonlinearity (1.1) propagates waves in both space and time with wavelength at $\mathcal{O}(1)$ and the wave speed in space is also at $\mathcal{O}(1)$. It is well known that the NLSW (1.1) conserves the mass [1,2]

$$N(t) := \int_{\mathbb{T}^d} |\psi(\mathbf{x}, t)|^2 d\mathbf{x} - 2\alpha \int_{\mathbb{T}^d} \mathrm{Im}\left(\overline{\psi(\mathbf{x}, t)}\partial_t \psi(\mathbf{x}, t)\right) d\mathbf{x} \equiv N(0), \quad t \ge 0,$$

and the energy

$$E(t) := \int_{\mathbb{T}^d} \left[\alpha|\partial_t \psi(\mathbf{x}, t)|^2 + |\nabla\psi(\mathbf{x}, t)|^2 + \frac{\varepsilon^{2p}}{p+1}|\psi(\mathbf{x}, t)|^{2p+2}\right] d\mathbf{x} \equiv E(0), \quad t \ge 0,$$

where $\overline{c}$ and Im(c) denote the conjugate and imaginary part of c, respectively.

The NLSW arises from different physical fields including the nonrelativistic limit of the Klein-Gordon equation [26,27,29], the Langmuir wave envelope approximation in plasma [8,12], and the modulated planar pulse approximation of the sine-Gordon equation for light bullets [5,33]. In the past decades, the NLSW (1.1) with $\varepsilon = 1$ and $0 < \alpha \ll 1$ has been widely studied analytically and numerically [1,2,8,26,27]. Along the analytical front, the existence of the solution and the convergence rate to the nonlinear Schrödinger equation (NLSE) have been investigated [8,26,27,29]. In the numerical aspect, different efficient numerical methods have been proposed and the conservative finite difference methods are most popular [1, 10, 13, 19, 31, 34]. In particular, the exponential wave integrator sine pseudospectral (EWI-SP) method has been proposed with optimal uniform error bounds in time established rigorously [2]. For more details related to the numerical schemes, we refer to [9,20,22,24,30,32,35] and references therein.

Moreover, rescaling the amplitude of the wave function $\psi(\mathbf{x}, t)$ by introducing a new variable $\phi := \phi(\mathbf{x}, t) = \varepsilon\psi(\mathbf{x}, t)$, the NLSW (1.1) can be reformulated as the following NLSW with $\mathcal{O}(1)$-nonlinearity and $\mathcal{O}(\varepsilon)$-initial data:

$$\begin{aligned} &i\partial_t \phi - \alpha\partial_{tt}\phi + \nabla^2\phi - |\phi|^{2p}\phi = 0, && \mathbf{x} \in \mathbb{T}^d, \quad t > 0, \\ &\phi(\mathbf{x}, 0) = \varepsilon\psi_0(\mathbf{x}), \quad \partial_t\phi(\mathbf{x}, 0) = \varepsilon\psi_1(\mathbf{x}), && \mathbf{x} \in \mathbb{T}^d. \end{aligned} \tag{1.2}$$

The long-time dynamics of the NLSW with $\mathcal{O}(\varepsilon^{2p})$-nonlinearity and $\mathcal{O}(1)$-initial data, i.e. the NLSW (1.1), is equivalent to that of the NLSW with $\mathcal{O}(1)$-nonlinearity and $\mathcal{O}(\varepsilon)$-initial data, i.e. the NLSW (1.2).

In recent years, long-time dynamics of dispersive partial differential equations (PDEs) including the (nonlinear) Schrödinger equation, nonlinear Klein-Gordon equation and Dirac equation with weak nonlinearity or small potential are thoroughly studied in the literature [3,4,7,15–17]. Exponential wave integrators and time-splitting methods are widely