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An Efficient Numerical Solution Method for Elliptic Problems in Divergence Form

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Abstract. In this paper the problem $-\text{div}(a(x,y)\nabla u)=f$ with Dirichlet boundary conditions on a square is solved iteratively with high accuracy for u and ∇u using a new scheme called "hermitian box-scheme". The design of the scheme is based on a "hermitian box", combining the approximation of the gradient by the fourth order hermitian derivative, with a conservative discrete formulation on boxes of length $2h$. The iterative technique is based on the repeated solution by a fast direct method of a discrete Poisson equation on a uniform rectangular mesh. The problem is suitably scaled before iteration. The numerical results obtained show the efficiency of the numerical scheme. This work is the extension to strongly elliptic problems of the hermitian box-scheme presented by Abbas and Croisille (J. Sci. Comput., 49 (2011), pp. 239–267).

AMS subject classifications: 65N35, 65N08

Key words: Hermitian scheme, box-scheme, Kronecker product, fast solver, iterative method, Poisson problem.

1 Introduction

Many references like Collatz [1], Forsythe and Wasow [2], Mitchell and Griffiths [3] and Iserles [4] treat the numerical resolution of partial differential equations as an educative building block in Applied Mathematics and Scientific Computing. For recent works, we refer to [5–11]. Beyond the design of specific numerical schemes which deals with accuracy and stability, the need of an efficient fast solver is a crucial issue to perform practical computations. The use of such solvers in canonical geometries remains at the heart of many computing codes in physics. Examples are among others fluid dymanics (compressible or incompressible Navier-Stokes equations), [12–14], the Helmholtz equation [15], computations in astrophysics, [16] or in geophysics, [17]. The scheme referred

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as *hermitian box-scheme* was first introduced in [18]. It combines a finite volume "box" approach with a hermitian computation of the derivative. The practical resolution is performed by a direct resolution algorithm using the Sherman-Morrison formula based on the Fast Fourier Transform following previous works like [19], or [20] in a different context. This approach proves to be very efficient from the computing point of view. In [21] theoretical and numerical studies of the hermitian box-schemes using finite element methods is given. The main properties of the hermitian box-schemes compared to other methods like finite element method are the fourth order accuracy for u and ∇u on regular problems, very good capability to handle sharp contrast in the diffusion coefficients and a great flexibility in the design permitted by the variation of the quadrature rule for the gradient. In [22], a new hermitian box-scheme in one dimension (called HB-scheme) is introduced and analysed with approximations of order 1 of the derivatives on the boundary. This scheme is applied to solve regular elliptic problems and elliptic problems with high contrast in ellipticity. The rate of convergence varies between 1 and 2.5 according to the regularity of the problem. In [23], we have introduced a new fourth order compact scheme on a cartesian grid for the Poisson problem in a square, whose design is based on the preliminary work [22]. As the approximations of the derivatives on the boundary is raised to order three (instead of order 1 in [22]), the HB-scheme appears numerically to be fourth order accurate for u and ∇u.

We have also introduced a fast solver (called HB-solver) based on the Sherman-Morrisson formula and Fast Fourier Transform. It is proved that HB-solver is of complexity $\mathcal{O}(N^2\log_2(N))$, where N is the number of collocation points.

Our motivation is to use the HB-scheme and the HB-solver to solve more complicated problems. The problems that are considered are nonseparable elliptic problems in the form

$$\begin{cases} -\mathrm{div}(a(x,y)\nabla u)=f & \text{on } \Omega=(a,b)^2, \\ u=g & \text{on } \bar{\Omega}, \end{cases} \tag{1.1}$$

with Dirichlet boundary conditions. The outline of this paper is as follows. In Section 2, we give the notations and we descirbe the principles of the scheme on the 2D Poisson problem, then we extend this scheme to nonseparable elliptic problem (1.1). In Section 3, we present in details the basis of Concus-Golub's algorithm and the iterative procedure combined with the fast Poisson solver of [23]. In Section 4, we focus on numerical tests in 2D. We observe a remarkable superconvergence of the solution and its gradient for regular problems.

2 Principle of the Hermitian box-scheme in two dimensions

This section is devoted to the principle of the hermitian box-scheme (HB-Scheme) in two dimensions. We start by summarizing the finite difference and matrix notations, then we recall the matrix form of the HB-scheme for the Poisson problem in two dimensions [23] and we give the matrix form for nonseparable elliptic problems.