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Rough Singular Integral Operators along Submanifolds

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Abstract. This paper is a summary of the research on the mapping properties of singular integrals with rough kernels and the corresponding maximal operators. More precisely, the author presents some recent progress, interesting problems and typical methods used in the theory concerning the boundedness and continuity for the rough singular integral operators and maximal singular integral operators along certain submanifolds such as polynomial mappings, polynomial curves, homogeneous mappings, surfaces of revolution and real-analytic submanifolds on the Lebesgue spaces, Triebel–Lizorkin spaces, Besov spaces and mixed radial-angular spaces.

Key Words: Singular integral, maximal singular integral, rough kernel, submanifolds.

AMS Subject Classifications: 42B20, 42B25

1 Background

Singular integral theory is one of the most important topics in harmonic analysis. On one hand, the classical Calderón–Zygmund singular integral operator derives from the Cauchy integral theory. The famous Hilbert transform is the prototype of all singular integrals, which is originated from researches of boundary value of conjugate harmonic functions on the upper half-plane. Actually, the Hilbert transform plays a key role in the convergence of Fourier integrals on the line. On the other hand, the Calderón–Zygmund singular integral operator derives from the study on the regularity of solution of second order elliptic equation with constant coefficients (see [12]). The Riesz transform is the higher dimensional example of classical Calderón–Zygmund singular integral operator. It is well known that the problem of regularity of solution for the Poisson equation is transferred to that of boundedness of the Riesz transform. Hence, the Calderón–Zygmund singular integral operators are nowadays intimately connected with PDEs, operator theory, several complex variables, and other fields.

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Generally, there are two kinds of singular integral operators: ones of convolution type and ones of nonconvolution type. Here we focus on the singular integrals of convolution type. As an extension of Calderón–Zygmund theory, singular Radon transforms along the appropriate submanifolds have been intensively studied. For relevant results and their applications one may consult lots of works of these mathematicians, such as M. Christ, D. Müller, E. M. Stein, D. H. Phong, F. Ricci, S. Wainger and so on. In this report, we shall mainly summarize some recent progress, interesting problems and typical methods used in the theory concerning the boundedness and continuity for the rough singular integral operators and maximal singular integral operators along the appropriate submanifolds on the Lebesgue spaces, Triebel–Lizorkin spaces, Besov spaces and mixed radial-angular spaces. The appropriate submanifolds include polynomial mappings, polynomial curves, homogeneous mappings, surfaces of revolution and real-analytic submanifolds.

In this section we first recall the related results on the rough singular integral operator along diagonal case as well as maximal singular integral operator.

1.1 Singular integral operator

Let Ω be homogeneous of degree zero, integrable over S^{n-1} and satisfy

$$\int_{\mathrm{S}^{n-1}} \Omega(y)d\sigma(y) = 0. \tag{1.1}$$

Let h be a suitable function defined on $\mathbb{R}_+$. The singular integral operator $T_{h,\Omega}$ is defined by

$$T_{h,\Omega}f(x) := \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)h(|y|)}{|y|^n} f(x-y)dy, \tag{1.2}$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz class on $\mathbb{R}^n$). When $h \equiv 1$, the operator $T_{h,\Omega}$ is the classical singular integral operator T_Ω.

Singular integral theory was initiated in the seminal work of Calderón and Zygmund [12]. The study of boundedness of rough singular integrals of convolution type has been an active area of research since the middle of the twentieth century. There are many rich and significant results about the boundedness of singular integral operators with rough kernels on the Lebesgue spaces, see, e.g., [13, 19, 21–24, 31, 32, 44, 71, 76]. The first work was due to Calderón and Zygmund [13] who used the Calderón–Zygmund rotation method (CZ-method for short) to establish the $L^p(\mathbb{R}^n)$ bounds of T_Ω for $1 < p < \infty$ if $\Omega \in L\log^+ L(\mathrm{S}^{n-1})$. Here $L(\log^+ L)^\alpha(\mathrm{S}^{n-1})$ for $\alpha > 0$ is the set of all measurable functions Ω on S^{n-1} which satisfy:

$$\|\Omega\|_{L(\log^+ L)^\alpha(\mathrm{S}^{n-1})} := \int_{\mathrm{S}^{n-1}} |\Omega(\theta)| \log^\alpha(2 + |\Omega(\theta)|)d\sigma(\theta) < \infty.$$