Analysis in Theory and Applications
Anal. Theory Appl., Vol. **39**, No. 2 (2023), pp. 178-190

DOI: 10.4208/ata.OA-2021-0013
June 2023

Hardy Operators and Commutators on Weighted Herz Spaces

Jingling Hu, Yangke Peng and Wenming Li*

School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, Hebei 050024, China

Received 8 May 2021; Accepted (in revised version) 26 June 2022

Abstract. Let P be the classical Hardy operator on $(0,\infty)$ and Q be the adjoint operator. In this paper, we get the boundedness for P, Q and the commutators of P and Q with CMO functions on the weighted Herz spaces.

Key Words: Hardy operator, commutator, CMO, weighted Herz space.

AMS Subject Classifications: 42B20, 42B25

1 Introduction

Let P and Q be the classical Hardy operator and its adjoint on $\mathbb{R}^+ = (0,+\infty)$,

$$Pf(x) = \frac{1}{x}\int_0^x f(y)dy, \quad Qf(x) = \int_x^\infty \frac{f(y)}{y}dy, \quad x > 0.$$

Hardy [4,5] established the Hardy integral inequalities

$$\|Pf\|_{L^p(\mathbb{R}^+)} \le p'\|f\|_{L^p(\mathbb{R}^+)}, \quad \|Qf\|_{L^p(\mathbb{R}^+)} \le p\|f\|_{L^p(\mathbb{R}^+)},$$

where $p > 1$ and $p' = p/(p-1)$.

The two inequalities above go by the name of Hardy's integral inequalities. For earlier development of this kind of inequality and many applications in analysis, see [6,8,13].

For $1 < p < \infty$, we say a weight w satisfies the $A_{p,0}$ condition, denoted as $w \in A_{p,0}$, if

$$[w]_{p,0} = \sup_{t>0} \frac{1}{t}\int_0^t w(y)dy\Big(\frac{1}{t}\int_0^t w(y)^{-p'/p}dy\Big)^{p/p'} < \infty.$$

*Corresponding author. *Email addresses:* `1209844033@qq.com` (J. Hu), `591383807@qq.com` (Y. Peng), `lwmingg@sina.com` (W. Li)

For $p=1$, we say a weight w satisfies the $A_{1,0}$ condition, denoted as $w\in A_{1,0}$, if for any $t>0$,

$$\frac{1}{t}\int_0^t w(y)dy\le Cw(x),\quad \text{a.e. } x\in(0,t).$$

Duoandikoetxea, Martin-Reyes and Ombrosi in [2] introduced the $A_{p,0}$ weight condition and proved that for $w\in A_{p,0}$,

$$\|Pf\|_{L^p(w)}\le C\|f\|_{L^p(w)},\quad \|Qf\|_{L^p(w)}\le C\|f\|_{L^p(w)}.$$

They also obtained that for $w\in A_{1,0}$, Hardy operator P and its adjoint operator Q are bounded from $L^1(w)$ to $L^{1,\infty}(w)$.

Let $b\in L_{loc}(\mathbb{R}^+)$, the commutators of Hardy operator P and its adjoint Q are defined by

$$\begin{aligned}P_bf(x)&=b(x)Pf(x)-P(bf)(x),\\ Q_bf(x)&=b(x)Qf(x)-Q(bf)(x).\end{aligned}$$

The spaces CMO^p were introduced, in the one-dimensional case, by Chen and Lau [1].

Let $1\le p<\infty$, we say that $b\in CMO^p(\mathbb{R}^+)$, if

$$\|b\|_{CMO^p}=\sup_{r>0}\Big(\frac{1}{r}\int_0^r|b(y)-b_{(0,r]}|^pdy\Big)^{1/p}<\infty,$$

where

$$b_{(0,r]}=\frac{1}{r}\int_0^r f(x)dx.$$

By the definition of CMO function. It is easy to see

$$\mathrm{CMO}^q(\mathbb{R}^+)\subsetneqq\mathrm{CMO}^p(\mathbb{R}^+)$$

for $1\le p<q<\infty$.

Long and Wang [11] established the Hardy's integral inequalities for commutators generated by P and Q with CMO function. Li, Zhang and Xue in [9] obtained some two-weight inequalities for commutators generated by P and Q with CMO function.

Herz spaces on $\mathbb{R}^n$ are defined by Herz in [7]. The weighted Herz spaces on $\mathbb{R}^n$ are defined by Lu and Yang in [12]. Fu, Liu, Lu and Wang [3] obtain the boundedness for n-dimensional Hardy operator on the Herz spaces.

The Herz spaces on the spaces of homogeneous type are defined by Liu and Zeng in [10]. Notice that $\mathbb{R}^+$ is a space of homogeneous type with distance $d(x,y)=|x-y|$ and Lebesgue measure, we define the weighted Herz spaces on $\mathbb{R}^+$ as following.

Let $B_k=(0,2^k]$, $C_k=B_k\backslash B_{k-1}$, $\chi_k=\chi_{C_k}$, $k\in\mathbb{Z}$, where χ_I is the characteristic function of set I. Let $\alpha\in\mathbb{R}$, $0<p\le\infty$, $0<q<\infty$. For the nonnagetive measurable function w on $\mathbb{R}^+$, the homogenous weighted Herz spaces $K_q^{\alpha,p}(w)$ is defined by

$$K_q^{\alpha,p}(w)=\big\{f\in L^q_{loc}(w):\|f\|_{K_q^{\alpha,p}(w)}<\infty\big\},$$