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The Boundedness of Marcinkiewicz Integrals on Grand Variable Herz Spaces

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Abstract. Under some natural regularity assumptions on the exponent function, we obtain the boundedness of Marcinkiewicz integrals μ, μ_S^ρ and $\mu_\lambda^{*,\rho}$ on grand variable Herz spaces. Our results enrich and improve some previous results in the literature.

Key Words: Marcinkiewicz integrals, Herz spaces, variable exponent, grand spaces.

AMS Subject Classifications: 46E30, 42B35

1 Introduction

Suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma(x')$. Let $\Omega \in L^1(\mathbb{S}^{n-1})$ be homogeneous of degree zero and satisfy

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$. Then the n-dimension Marcinkiewicz integral operator μ is defined by

$$\mu(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}},$$

where

$$F_t(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

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Corresponding to the parameterized Lusin area integral S^ρ and the Littlewood-Paley $g_\mu^{*,\rho}$ function, the Marcinkiewicz integrals μ_S^ρ and $\mu_\lambda^{*,\rho}$ are defined by

$$\mu_S^\rho(f)(x) = \left(\int\int_{\Gamma(x)} \left| \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}},$$

$$\mu_\lambda^{*,\rho}(f)(x) = \left(\int\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \left| \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}}, \quad \lambda > 1,$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}$, $\rho > 0$.

In 1958, Stein [29] first introduced the operator μ and proved that μ is of type (p,p) for $1 < p \le 2$ and of weak type $(1,1)$ in the case of $\Omega \in Lip_\gamma(\mathbb{S}^{n-1})$ $(0 < \gamma \le 1)$. In 1962, Benedek, Calderón and Panzone [4] extended Stein's result in the case of $\Omega \in C^1(\mathbb{S}^{n-1})$ and $1 < p < \infty$. In 1972, Walsh [32] got the similar result using a rough kernel instead of the smooth kernel. In 2000, Ding, Fan and Pan [7] improved the above results in the case of $\Omega \in H^1(\mathbb{S}^{n-1})$ (the Hardy space on $\mathbb{S}^{n-1}$). In 2002, Al-Salman et al. [3] extended Walsh's result with $\Omega \in L(\log^+ L)^{\frac{1}{2}}(S^{n-1})$. Ding, Liu and Xue [9] gave the boundedness of Marcinkiewicz integrals on Hardy spaces. Subsequently, a considerable amount of attention has been given to study the boundedness of this operator, see e.g., [8,9,14,25,26,31,34,35,38] for more details.

Since the paper [19] by Kováčik and Rákosník appeared in 1991, the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ have been extensively investigated. Boundedness of Calderón-Zygmund operators, fractional integrals and Marcinkiewicz integrals, etc. on $L^{p(\cdot)}(\mathbb{R}^n)$ have been obtained in [2,6,10,11,33]. Herz spaces with variable exponent have been recently introduced in [1,15,16,18,30,35–37]. In 2012, Wang, Fu and Liu [33] proved that Marcinkiewicz integrals were bounded on the Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. In 2016, Liu and Wang [24] extended the $L^{p(\cdot)}(\mathbb{R}^n)$ boundedness of Marcinkiewicz integrals to the variable exponent Herz spaces $\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$.

In recent years, Grand Lebesgue spaces on bounded sets have been widely studied, see [5, 13, 17, 28]. Kokilashvili and Meskhi [20–23] gave the boundedness of various operators of harmonic analysis in grand Lebesgue spaces. Grand Lebesgue sequence spaces were introduced in [28], where maximal, convolutions, Hardy, Hilbert and fractional operators were studied in these spaces. In [27], the authors introduced grand variable Herz space $\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)$ and obtained the boundedness of sublinear operators on $\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)$. Motivated by the results mentioned above, the aim of this paper is to study Marcinkiewicz integrals on grand variable Herz spaces.

In general, different constant in the same series of inequalities will mainly be denoted by c or C. $f \approx g$ means that $c_1 f \le g \le c_2 f$ if there exist constants c_1, c_2.