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Weak Harnack Inequalities for Eigenvalues and the Monotonicity of Hessian's Rank

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Abstract. We study microscopic convexity properties of convex solutions of fully nonlinear parabolic equations under a structural condition introduced by Bian-Guan. We prove weak Harnack inequalities for the eigenvalues of the spatial Hessian of solutions and obtain the monotonicity of Hessian's rank with respect to time.

Key Words: Harnack inequalities, parabolic equations, microscopic convexity.

AMS Subject Classifications: 35K55, 35E10

1 Introduction

The convexity of solutions is an important topic in the study of partial differential equations, and there are two main research methods: macroscopic methods and microscopic methods.

For the macroscopic convexity argument, Korevaar initially established a concavity maximum principle for quasilinear equations in [13, 14]. This result was used by Kennington in [12] to prove that for a class of parabolic equations the level sets of solutions are convex. Later, Korevaar's result was improved for parabolic equations by Greco-Kawohl in [9]. Recently, Juutinen extended it to viscosity solutions of certain fully nonlinear parabolic equations in [11].

The microscopic convexity is concerned about Hessian's ranks of solutions. The microscopic technique for the convex solution was first established by Caffarelli-Friedman in [4] and Yau in [16] at the same time and then it was extended to high dimensions by Korevaar-Lewis [15]. Later in [1,2,5,7,8] it was generalized to fully nonlinear elliptic and parabolic equations. One method to establish microscopic convexity is to introduce the elementary symmetric polynomials σ_k of eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

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of the Hessian as an auxiliary function. Recently, Székelyhidi-Weinkove in [18,19] gave a new method for nonlinear elliptic equations by using a simple linear auxiliary function

$$\lambda_l + 2\lambda_{l-1} + \cdots + l\lambda_1. \tag{1.1}$$

This method utilizes the concavity of sums of the lowest eigenvalues.

Bian-Guan in [1] considered solutions of nonlinear parabolic equations

$$u_t = F(D^2u, Du, u, x, t)$$

under a structural condition for F (see (1.2) below), proved a constant rank theorem for a fixed time and the monotonicity of the rank with respect to time by using the auxiliary function $\sigma_{l+1} + \frac{\sigma_{l+2}}{\sigma_{l+1}}$. In this paper, we make the same assumption on F as in [1]. Based on the approach of [19] and using again the auxiliary function (1.1), we directly prove the weak Harnack inequality for each eigenvalue λ_i and the monotonicity of the rank with respect to time is a direct corollary of the weak Harnack inequalities.

Now we state our results precisely. For $\theta, R, \varepsilon > 0$, we define

$$\begin{aligned} &Q = Q(\theta, R) = \{(t,x) \in \mathbb{R}^{n+1} |\ t \in (0, \theta R^2),\ |x| < R\}, \\ &Q_\varepsilon(\theta, R) = \{(t,x) \in \mathbb{R}^{n+1} |\ t \in (\varepsilon, \theta R^2 - \varepsilon),\ |x| < R - \varepsilon\}. \end{aligned}$$

Let $Sym_n^+(\mathbb{R})$ denote the space of semi-positive definite $n \times n$ matrices and F be a function

$$F = F(A, p, u, x, t) \in C^2(Sym_n^+(\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R} \times Q).$$

We assume that F satisfies the structural condition in [1] that

$$F(A^{-1}, p, u, x, t) \text{ is locally convex in } (A, u, x) \text{ for each pair } (p, t). \tag{1.2}$$

Suppose that $u \in C^3(Q)$ is a convex solution of

$$u_t = F(D^2u, Du, u, x, t), \tag{1.3}$$

where D^2u denotes the spatial Hessian $(u_{x_ix_j})$, $Du = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$, and F satisfies the elliptic condition that for all $\xi \in \mathbb{R}^n$

$$\Lambda^{-1}|\xi|^2 \le F^{ij}(D^2u, Du, u, x, t)\xi^i\xi^j \le \Lambda|\xi|^2 \quad \text{on } Q, \tag{1.4}$$

for a constant $\Lambda > 0$, where F^{ij} is the derivative of F with respect to the (i,j)th entry A_{ij} of A. Our main result is as follow.

Theorem 1.1. *Let u be as above and $0 \le \lambda_1 \le \cdots \le \lambda_n$ be eigenvalues of the spatial Hessian D^2u. Let $\varepsilon > 0$, $0 \le \theta_1 < \theta_2 < \theta$, $r_1, r_2 \in (0,1)$, $0 < R \le 1$,*

$$Q_\varepsilon = Q_\varepsilon(\theta, R), \quad Q_\varepsilon^1 = Q_\varepsilon\left(\frac{\theta_1}{r_1^2}, r_1 R\right), \quad Q_\varepsilon^2 = (\theta_2 R^2, 0) + Q_\varepsilon\left(\frac{\theta - \theta_2}{r_2^2}, r_2 R\right),$$