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Extension of the ENO-ET Reconstruction Scheme to Two Space Dimensions on Cartesian Meshes in Conjunction with the ADER Approach

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Dedicated to Professor Tao Tang on the occasion of his 60th birthday.

Abstract. Godunov's Theorem [S.K. Godunov, *Mat. Sb.* **47** (1959)], stated more than six decades ago, set the framework for understanding the limitations of linear numerical schemes for approximating hyperbolic equations numerically. This theoretical result sets one of the basic requirements for constructing high-order numerical schemes, namely non-linearity. In the present article we are concerned with modifications to essentially-non-oscillatory (ENO) non-linear reconstruction approach, along with fully discrete ADER schemes to derive methods of arbitrary order of accuracy in space and time. Here we extend a recently proposed ENO-ET scheme for one-dimensional problems to two space dimensions with Cartesian meshes. The methods are implemented up to fifth order of accuracy and assessed via three scalar 2D problems, namely the linear advection equation, Burgers equation and a kinematic frontogenesis model used in meteorology. Empirical convergence rates are studied for the classical ENO, classical WENO and the newly proposed ENO-ET. For smooth solutions results from the newly proposed ENO-ET reconstruction scheme are superior to those of conventional ENO in terms of theoretically expected convergence rates and size of errors. Compared to the results obtained with WENO reconstruction, the performance of ENO-ET for second and third orders is superior. For discontinuous solutions, again ENO-ET is superior, in that it captures wave amplitudes more accurately than ENO as accurate as WENO and, unlike ENO, exhibits no spurious oscillations near discontinuities.

AMS subject classifications: 65M08, 76M12

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1. Introduction

Godunov's theorem [13] sets the framework for understanding the limitations of linear numerical schemes for approximating hyperbolic equations numerically. This theoretical re-

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sult also sets one of the basic requirements for constructing high-order numerical schemes, namely non-linearity. There are two main approaches to construct non-linear schemes to approximate hyperbolic equations the total variation diminishing (TVD) methods pioneered by Kolgan [20] (see also [34, 42], and the schemes based on non-linear spatial reconstructions pioneered by Harten and Osher [18] (see also [16, 17]). In the present article we are concerned with numerical schemes based non-linear reconstructions. For decades, the essentially-non-oscillatory (ENO) approach [18] has been one of the key approaches to generate non-linear reconstruction procedures. Interpolations can be derived from point-wise samples or cell integral averages, the latter being more suitable for finite volume schemes. It is well known that interpolations can suffer from the Runge phenomenon [21]. Piecewise interpolation overcomes the Runge phenomenon by defining local and independent polynomials for each cell. ENO reconstructions use an adaptive stencil coupled to a local smoothness criterion to avoid, as much as possible, interpolations across discontinuities. Local truncation error analysis allows us to formally verify that the ENO reconstruction is uniformly high-order and can also capture sharp discontinuities with much reduced spurious oscillations. However, there is no closed convergence theory for ENO schemes and analysis of the properties of the methods has to be done case by case. In this sense, there exists evidence that the ENO reconstruction procedure fails, even for the linear advection equation with smooth initial condition. For instance Rogerson and Meiburg [28] have reported the lack of accuracy for the initial condition $q(x,0) = e^{-x}$. A similar failure was detected by Shu [31] for $q(x,0) = \sin^4(\pi x)$. The source of the difficulties in both cases is the selection of the linearly unstable downwind stencil which causes a switching of stencils and avoid error cancellations in conservative schemes. In [31], Shu proposed a modification of ENO consisting of changing the stencil selection criterion, such that this is biased to the central stencil. The modification recovers the expected theoretical order of accuracy. A successful way of escaping from the limitations of ENO is the weighted essentially non-oscillatory (WENO) approach pioneered by Jiang and Shu [19] (see also [8, 22]). In the present work we stay with ENO and ENO-type schemes, as these are relatively simple as compared to WENO schemes.

In recent years, various modifications to the ENO reconstruction scheme has been proposed, including the already mentions mENO of Shu [31]. Fu and collaborators [11, 12] also proposed a family of ENO-type schemes, called targeted ENO (TENO). Another ENO-type scheme is the averaged ENO reconstruction (AENO) reported in [38], where two neighbouring polynomials are averaged the ENO polynomial and that from the neighbouring stencil in the central direction. A recently proposed ENO-type reconstruction procedure ENO-ET [27], is based on extended stencils obtained from the ENO search. Numerical evidence for the one-dimensional case [27] shows that ENO-ET can attain the theoretically expected convergence rates for high-order schemes applied to the linear advection and the Euler equations with smooth profiles. The schemes has also overcome the lack of accuracy of ENO for the initial condition $\sin^4(\pi x)$.

Reconstruction procedures can be employed in semidiscrete approaches [3, 32, 33] or fully discrete approaches, such as those of the family of ADER (Arbitrary DERivative Riemann problem approach) [9, 23, 29, 35, 36, 39, 41]. In the ADER schemes, the spatial and