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Adaptive High-Order A-WENO Schemes Based on a New Local Smoothness Indicator

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Dedicated to Professor Tao Tang on the occasion of his 60th birthday.

Abstract. We develop new adaptive alternative weighted essentially non-oscillatory (A-WENO) schemes for hyperbolic systems of conservation laws. The new schemes employ the recently proposed local characteristic decomposition based central-upwind numerical fluxes, the three-stage third-order strong stability preserving Runge-Kutta time integrator, and the fifth-order WENO-Z interpolation. The adaptive strategy is implemented by applying the limited interpolation only in the parts of the computational domain where the solution is identified as rough with the help of a smoothness indicator. We develop and use a new simple and robust local smoothness indicator (LSI), which is applied to the solutions computed at each of the three stages of the ODE solver. The new LSI and adaptive A-WENO schemes are tested on the Euler equations of gas dynamics. We implement the proposed LSI using the pressure, which remains smooth at contact discontinuities, while our goal is to detect other rough areas and apply the limited interpolation mostly in the neighborhoods of the shock waves. We demonstrate that the new adaptive schemes are highly accurate, non-oscillatory, and robust. They outperform their fully limited counterparts (the A-WENO schemes with the same numerical fluxes and ODE solver but with the WENO-Z interpolation employed everywhere) while being less computationally expensive.

AMS subject classifications: 65M06, 76M20, 76N15, 76L05, 35L65

Key words: Local smoothness indicator, scheme adaption, strong stability preserving Runge-Kutta methods, hyperbolic systems of conservation laws, A-WENO schemes.

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1. Introduction

This paper focuses on developing high-order finite-difference methods for hyperbolic systems of conservation laws. We consider one-dimensional (1-D),

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}, \tag{1.1}$$

and two-dimensional (2-D),

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x + \boldsymbol{G}(\boldsymbol{U})_y = \boldsymbol{0}, \tag{1.2}$$

systems, though the proposed techniques can be directly extended to higher-dimensional cases. Here, x and y are spatial variables, t is the time, $\boldsymbol{U} \in \mathbb{R}^d$ is a vector of unknown functions, and $\boldsymbol{F} : \mathbb{R}^d \to \mathbb{R}^d$ and $\boldsymbol{G} : \mathbb{R}^d \to \mathbb{R}^d$ are nonlinear fluxes.

It is well-known that solutions of (1.2) may develop complicated wave structures, including shocks, rarefactions, and contact discontinuities, even when the initial data are infinitely smooth. Therefore, it is challenging to develop highly accurate and robust numerical methods for (1.2). We refer the reader to various existing numerical methods, including high-order ones, e.g., the monographs and review papers [8, 30, 35, 40, 55, 56, 60] and references therein.

Semi-discretization of (1.1) and (1.2) offers one of the popular frameworks for constructing high-order finite-volume and finite-difference schemes: the spatial derivatives are approximated using appropriate numerical fluxes. At the same time, the time evolution is conducted with the help of a high-order and stable ODE solver. To achieve a high order of spatial accuracy, the numerical fluxes must be evaluated using the point values of $\boldsymbol{U}$ obtained by an appropriate piecewise polynomial reconstruction (interpolation) of the computed solution. In order to enforce nonlinear stability, the reconstructions have to employ nonlinear limiters designed to prevent spurious oscillations in the nonsmooth parts of the solutions. Popular finite-volume reconstructions, such as essentially non-oscillatory (ENO) (see, e.g., [1, 28, 29, 56]) and weighted ENO (WENO) (see, e.g., [6, 31, 44, 55, 56]) ones are highly accurate, but typically finite-volume ENO and WENO schemes are computationally expensive, especially in the multidimensional case. More efficient implementations of ENO and WENO reconstructions can be carried out within the finite-difference framework in a dimension-by-dimension manner; see, e.g., [7, 11, 12, 31, 57, 58]. Unfortunately, the finite-difference schemes, which are directly based on finite-volume reconstructions, rely on flux splittings, substantially increasing the amount of numerical diffusion present in finite-volume ENO and WENO schemes. This drawback of finite-difference WENO schemes was overcome in [32] (also see [43]), where alternative WENO (A-WENO) schemes were introduced. A-WENO schemes employ standard finite-volume numerical fluxes (without any need for flux splitting and related modifications), whose accuracy, in the context of finite-difference schemes, is limited to the second order, while a high order is achieved using the flux Taylor expansion and high-order WENO-Z interpolations, which were developed in [16, 21, 32, 43, 64]. For several recent A-WENO schemes based on different finite-volume numerical fluxes, we refer the reader to [62–64].