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A Novel Approach to Simulate Lane-Emden and Emden-Fowler Equations using Curriculum Learning-Based Unsupervised Symplectic Artificial Neural Network

Arup Kumar Sahoo and S. Chakraverty*

Department of Mathematics, National Institute of Technology Rourkela, Rourkela-769008, Odisha, India.

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Abstract. This paper investigates the impact of the curriculum learning process in a multilayer neural network (NN) for solving the Lane-Emden and Emden-Fowler models. Starting from the training of a neural network in a small domain, we gradually expanded the domain. The symplectic NN trial solution is used for solving titled models. Feedforward NN and error back-propagation algorithms are used to minimize the error function and modify the parameters. The consistency of the algorithm is demonstrated by solving several problems. Calculating different types of errors (MSE up to 1E-10), we show an excellent agreement between the current simulations and existing results.

AMS subject classifications: 34A12, 34A30, 34A34, 68T07, 85-08, 85-10

Key words: Curriculum learning, symplectic neural network, unsupervised, Lane-Emden equation, Emden-Fowler equation.

1. Introduction

At the turn of the twenty-first century, science and technology have been greatly governed by artificial intelligence (AI) techniques. The origin of AI may be traced back to the 1950s when John McCarthy organized a two-month workshop at the Dartmouth College. With the increase in global requirements, scientists around the world have been working towards the development of various novel and advanced AI techniques which are efficient and have less time complexity. Some of AI techniques include neural network or connectionist systems, experts systems, natural language processing, fuzzy logic, etc. Artificial neural networks (ANN) are a rapidly growing research area of AI, which revolutionise science and technology over the years.

*Corresponding author. *Email addresses:* `sne_chak@yahoo.com` (S. Chakraverty), `arupnitr.jrfmath@gmail.com` (A.K. Sahoo)

ANN has become the focus of many researchers owing to its growing applications in diverse fields such as chemical, physics, medicine, finance, trading etc. ANN models have placed immense demand in computing infrastructure. After the development of the back-propagation (BP) algorithm in the late 1980s, various dynamical problems have been handled by ANN, which may be difficult to solve by traditional methods. ANN is a self-adaptive and can learn from complex environments in both supervised and unsupervised methods. Another area where ANN has been proven effective in recent years is to find the solution of a singular system. In past decades a number of traditional numerical methods such as Adomian decomposition method (ADM), homotopy perturbation method (HPM), and finite difference method have been used to deal with singularity. These methods provide a satisfactory approximation to the solution for differential equations (DEs) [8]. However, ANN-based solutions have a number of benefits over traditional numerical approaches. Neural solutions are discrete in nature and analytic. Another exciting feature of the neural model is that, once the NN is well-established, it can be used as a black box. Like the human brain, neural networks perform better when we start simple and progressively increase the challenge level [13]. In the context of machine learning, it is referred as the curriculum learning [5].

In the pioneering work, Lee and Kang [25] developed a neural model using Hopfield NN for solving first-order DEs, Lagaris *et al.* [23] proposed a neural method to solve boundary value problems with irregular boundaries. He predicted solutions of DEs by generating a trial function that satisfies the initial/boundary conditions associated with it. For solving singular value problem (SVP) using NN; research was carried out by Tawfiq and Hussein [45]. Sahoo and Chakraverty [40] in a new work, proposed a novel unsupervised neural model for solving non-linear oscillatory systems. Many essential works on neural network model to solve DE has been done in past years, and different research papers had been composed by various authors [11, 27, 31].

The Lane-Emden and Emden-Fowler equations are a particular case of SVP frequently used in astrophysics and mathematical physics. The equation was first introduced by astrophysicist Lane [24] and further described in more detail by Emden. In current ages, the Lane-Emden model has played an essential role in various fields such as classical and quantum mechanics [35], dusty fluid models [15], governing polytropic and isothermal gas spheres [7], isotropic continuous media [37], stellar structure [44], catalytic diffusion reactions [34], electromagnetic theory [21], density profile of gaseous star [26] and other areas of mathematical physics [6]. For solving Lane-Emden equations, Shawagefeh [42] and Wazwaz [51] used the Adomian decomposition method (ADM) successfully. Wazwaz also used ADM for solving time-dependent Emden-Fowler type equations in his investigation [52]. In order to find the numerical and analytical solution of titled DE homotopy perturbation method (HPM) [12], rational Legendre pseudospectral method (RLM) [36], Haar wavelet approximate method [20], homotopy perturbation method with Laplace transform (LT-HPM) [46] has been developed in past years. Few more important papers on existing methods for solving Lane-Emden model are listed in Table 1.

From the above literature reviews, it reveals that a good number of numerical and analytical solvers have been introduced to find the approximate solution of dynamical prob-