East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2022-177.220922

Vol. **13**, No. 2, pp. 257-275
May 2023

Convergence of Extrapolated Dynamic String-Averaging Cutter Methods and Applications

Nguyen Buong[1,2,3,*] and Nguyen Duong Nguyen[4]

[1] *Institute of Theoretical and Applied Research, Hanoi 100000, Vietnam.*
[2] *Faculty of Information Technology, Duy Tan University, Da Nang 550000, Vietnam.*
[3] *Vietnam Academy of Science and Technology, Institute of Information Technology, 18, Hoang Quoc Viet, Hanoi, Vietnam.*
[4] *Foreign Trade University, 91, Chua Lang, Lang Thuong, Dong Da, Hanoi, Vietnam.*

Received 8 June 2022; Accepted (in revised version) 22 September 2022.

Abstract. Two extrapolated dynamic string-averaging cutter methods for finding a common fixed point of a finite family of demiclosed cutters in a Hilbert space are developed. One method converges weakly to a common fixed point of the family. The other converges in norm and is a combination of the method mentioned and the steepest-descent method. The proof of the strong convergence does not employ any additional cutter related conditions such as approximate shrinking and bounded regularity of their fixed point sets often used in literature. Particular cases of the last method and applications to a convex optimization problem over the intersection of the level sets and the LASSO problem with computational experiments are provided as illustrations.

AMS subject classifications: 46N10, 47H09, 47H10, 47J25, 47N10, 65F10, 65J99

Key words: Quasi-nonexpansive mapping, fixed point, variational inequality, steepest-descent method.

1. Introduction

Let H be a real Hilbert space equipped with the inner product $\langle .,. \rangle$ and with the corresponding norm $\|.\|$. Let $L := \{1,\dots,m\}$ with a fixed integer $m \geq 1$ and let T_i, for each $i \in L$, be a demiclosed cutter on H, satisfying the condition $\cap_{i\in L}\mathrm{Fix}(T_i) \neq \emptyset$ where

$$\mathrm{Fix}(T_i) := \{p \in H : p = T_i p\}$$

is the set of the fixed points of T_i. Note that in what follows, we often write C_i for $\mathrm{Fix}(T_i)$.

*Corresponding author. *Email addresses:* nguyenbuong@dtu.edu.vn,nbuong@ioit.ac.vn (Ng. Buong), duongnguyen@ftu.edu.vn (Ng.D. Nguyen)

The problem we consider in this paper consists in finding a point

$$p_* \in C := \cap_{i\in L} C_i. \tag{1.1}$$

When T_i is the metric projection P_{C_i} on a given closed convex subset C_i in H, the problem (1.1) is called the convex feasibility problem. Several iterative solution methods have been studied in [3, 5–19, 21, 35, 36]. When T_i is a nonexpansive mapping on H for each $i \in L$, Yamada [36] proposed the hybrid steepest-descent method,

$$x^{k+1} = (I - t_k \mu F) T x^k, \quad k \geq 0, \tag{1.2}$$

where I denotes the identity mapping in H, F is η-strongly monotone and l-Lipschitz continuous, $\mu \in (0, 2\eta/l^2)$ is a fixed number, the parameter t_k satisfies the conditions

(C1) $t_k \in (0,1)$ for all $k \geq 0$, $\lim_{k\to\infty} t_k = 0$ and $\sum_{k\geq 0} t_k = \infty$,

(C2) $\sum_{k\geq 0} |t_k - t_{k+m}| < \infty$ or $\lim_{k\to\infty} |t_k - t_{k+m}|/t_k = 0$,

and either

$$T = T_m T_{m-1} \cdots T_1 \quad \text{or} \quad T = \sum_{i\in L} \omega_i T_i \tag{1.3}$$

with $\omega_i > 0$ and $\sum_{i\in L} \omega_i = 1$. The first author and Duong [7] suggested a modification of (1.2), viz.

$$x^{k+1} = \left(1 - \beta_k^0\right) x^k + \beta_k^0 (I - t_k \mu F) T_m^k T_{m-1}^k \cdots T_1^k x^k, \tag{1.4}$$

where

$$T_i^k = I + \beta_k^i (T_i - I)$$

with $\beta_k^i \in [a, b] \subset (0,1)$ for all $k \geq 0, i \in L$ and

$$\lim_{k\to\infty} \left|\beta_{k+1}^i - \beta_k^i\right| = 0.$$

This method does not requires condition (C2) and converges strongly.

The problem (1.1) has been recently studied [6, 11, 22, 33]. Reich and Zalas [33] introduced the dynamic string-averaging method

$$x^{k+1} = T^k x^k, \quad T^k = \sum_{n=1}^{N_k} \omega_n^k S_n^k, \tag{1.5}$$

where $S_n^k = \prod_{i\in L_n^k} T_i$ is a product of mappings along the string $L_n^k \subset L$ for $n = 1, \ldots, N_k$, $\omega_n^k \in [\varepsilon, 1-\varepsilon]$ with $\sum_{n=1}^{N_k} \omega_n^k = 1$ and a sufficiently small value $\varepsilon > 0$. Under the following assumptions:

(A1) each mapping T_i is a demiclosed cutter,

(A2) $L \subseteq L_k \cup \cdots \cup L_{k+s-1}$ for some $s \geq m-1$ where $L_k = L_{N_k}^k$,