Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **15**, No. 4, pp. 1023-1055

DOI: 10.4208/aamm.OA-2021-0241
August 2023

The Formulation of Finite Difference RBFWENO Schemes for Hyperbolic Conservation Laws: An Alternative Technique

Rooholah Abedian[1,*] and Mehdi Dehghan[2]

[1] *School of Engineering Science, College of Engineering, University of Tehran, Iran*
[2] *Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Ave., Tehran, Iran*

Received 22 July 2021; Accepted (in revised version) 9 February 2022

Abstract. To solve conservation laws, efficient schemes such as essentially non-oscillatory (ENO) and weighted ENO (WENO) have been introduced to control the Gibbs oscillations. Based on radial basis functions (RBFs) with the classical WENO-JS weights, a new type of WENO schemes has been proposed to solve conservation laws [J. Guo et al., J. Sci. Comput., 70 (2017), pp. 551–575]. The purpose of this paper is to introduce a new formulation of conservative finite difference RBFWENO schemes to solve conservation laws. Unlike the usual method for reconstructing the flux functions, the flux function is generated directly with the conservative variables. Comparing with Guo and Jung (2017), the main advantage of this framework is that arbitrary monotone fluxes can be employed, while in Guo and Jung (2017) only smooth flux splitting can be used to reconstruct flux functions. Several 1D and 2D benchmark problems are prepared to demonstrate the good performance of the new scheme.

AMS subject classifications: 65M06, 35L65

Key words: Weighted essentially non-oscillatory scheme, radial basis functions interpolation, finite difference method, hyperbolic conservation laws.

1 Introduction

Recently, high resolution schemes have been extensively studied for hyperbolic systems of conservation laws. For example, as new high order schemes, RBFWENO schemes were created for hyperbolic conservation laws [10, 13, 18, 19]. RBFWENO methods in the form of finite differences to solve conservation laws were first introduced by Guo and Jung in [18]. The finite volume version of these schemes was introduced in [19]. RBFWENO

*Corresponding author.
Email: rabedian@ut.ac.ir (R. Abedian)

methods were also used to solve the Hamilton-Jacobi equations in [8, 9]. RBFWENO schemes were designed and developed based on successful schemes [22, 29, 32, 33].

In 1994, authors of [29] introduced the first third-order WENO schemes in finite volume form to solve one-dimensional equations of conservation laws. Later, authors of [22] introduced a general framework for constructing WENO schemes in finite difference form that are more efficient for multidimensional equations. Improvements to traditional WENO methods have been developed by various researchers, including WENO-M [21], WENO-Z [14], WENO-NS [20], WENO-η [16], WENO-P [25] and WENO schemes with adaptive order (WENO-AO). WENO-AO schemes started by Zhu and Qiu in [36] and was further developed by Balsara et al. [12]. Authors of [11, 17] constructed very high order WENO schemes. It should be noted that due to the high efficiency of WENO schemes, today these methods have been employed and expanded to solve other equations such as the Hamilton-Jacobi [1, 3, 7] and the degenerate parabolic equations [2, 4, 5, 23, 30].

Since the idea of reconstructing the flux functions introduced by Shu and Osher [33] is easy and clean to implement, most high order finite difference WENO schemes such as high order finite difference ENO [33], WENO [22] and RBFWENO [18] applied the same idea to reconstruct flux functions. However, the numerical flux with finite difference in flux formulation must have certain limitations in mathematical forms satisfying the forms of flux splitting, because non-linear stability and upwind are essential in schemes. Shu and Osher introduced another approach for constructing numerical fluxes in high order conservative finite difference schemes in [32]. Their idea was that interpolations are directly obtained on the point values of the solution rather than on the flux values. Jiang et al. introduced the combination of WENO schemes and Lax-Wendroff time discretization for solving conservation laws in [24]. They have used the idea introduced in [32] to reconstruct numerical fluxes. By applying the approach introduced in [32], the defects in classical schemes such as [18] mentioned above can be overcome. As mentioned in [24], although the use of this approach is very expensive compared to the standard one, but its main advantage is that arbitrary monotone fluxes can be considered in this framework, while the common method of reconstructing flux functions can only be used for smooth flux splitting.

A new type of finite difference radial basis functions WENO (RBFWENO) schemes for solving hyperbolic conservation laws was proposed by Guo and Jung in [18]. In this scheme, the reconstruction of fluxes is based on the common method of reconstructing flux functions. To increase the accuracy of the classical ENO/WENO, Guo and Jung focused on interpolation coefficients [18]. Infinitely smooth RBFs such as the multiquadratic RBF (MQ-RBF) and Gaussian RBF (G-RBF) which contain the shape parameter are adopted to improve local accuracy. The local accuracy can be increased by optimizing the shape parameter. It can be shown that when the shape parameter is zero, the RBF interpolation is equivalent to the polynomial interpolation, so by using the shape parameter we can take advantage of the fact that the polynomial interpolation is a special case of the RBF interpolation [26, 28]. The purpose of this paper is to study and develop the conservative finite difference RBFWENO schemes using other formulations to solve the