Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **15**, No. 4, pp. 901-931

DOI: 10.4208/aamm.OA-2021-0311
August 2023

A Coercivity Result of Quadratic Finite Volume Element Schemes over Triangular Meshes

Xueying Wen[1] and Yanhui Zhou[2,*]

[1] *School of Mathematics and Statistics, Northeast Normal University, Changchun, Jilin 130024, China*
[2] *School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou, Guangdong 510665, China*

Received 3 October 2021; Accepted (in revised version) 27 January 2022

Abstract. In this work, we study the coercivity of a family of quadratic finite volume element (FVE) schemes over triangular meshes for solving elliptic boundary value problems. The analysis is based on the standard mapping from the trial function space to the test function space so that the coercivity result can be naturally incorporated with most existing theoretical results such as H^1 and L^2 error estimates. The novelty of this paper is that, each element stiffness matrix of the quadratic FVE schemes can be decomposed into three parts: the first part is the element stiffness matrix of the standard quadratic finite element method (FEM), the second part is the difference between the FVE and FEM on the element boundary, while the third part can be expressed as the tensor product of two vectors. As a result, we reach a sufficient condition to guarantee the existence, uniqueness and coercivity result of the FVE solution on general triangular meshes. Moreover, based on this sufficient condition, some minimum angle conditions with simple, analytic and computable expressions are obtained. By comparison, the existing minimum angle conditions were obtained numerically from a computer program. Theoretical findings are conformed with the numerical results.

AMS subject classifications: 65N08

Key words: Quadratic finite volume element schemes, triangular meshes, coercivity result, minimum angle condition.

1 Introduction

Due to the local conservation property and other advantages, the finite volume method (FVM) is an important and popular numerical tool for solving partial differential equations (c.f. [17, 25]). The finite volume element method (FVEM) is a special member of

*Corresponding author.
Emails: 2581384589@qq.com (X. Wen), zhouyh9@mail2.sysu.edu.cn (Y. Zhou)

FVM, and many researchers pay a lot of attention to this method, see the book [18] and review papers [21,35]. On triangular meshes, the linear FVEM is closed to linear FEM, the stability and convergence analysis are studied in many works (e.g., [1,2,8,11–13,29,31]), the coercivity result does not require the minimum angle condition. In other words, linear FVEM is unconditionally stable on triangular meshes.

However, for the coercivity result of quadratic FVEM on triangular meshes, most existing works require a minimum angle condition. We first point out that, the existing quadratic FVE scheme of Lagrange type is constructed by two variable parameters α and β, where $\alpha \in (0,1/2)$ on the element boundary and $\beta \in (0,2/3)$ in the interior of element (c.f. [31]). Tian and Chen [24] and Liebau [20] studied the coercivity result of quadratic scheme $(\alpha,\beta)=(1/3,1/3)$ and $(\alpha,\beta)=(1/4,1/3)$, respectively. The theoretical results in [24] need the mesh assumption that the maximum angle of each triangular element is not greater than 90°, and the ratio of the lengths of the two sides of the maximum angle belongs to $[\sqrt{2/3},\sqrt{3/2}]$, while [20] required that the geometry of the triangulation triangles is not too extreme. A general construction of quadratic FVE schemes was presented in [31] by Xu and Zou, and they improved some existing coercivity results. For example, the minimum angle should be greater than or equal to 7.11° for the scheme proposed in [10] $(\alpha,\beta)=(1/6,1/4)$, 9.98° for the scheme proposed in [20] and 20.95° for the scheme proposed in [24]. Later, a general framework for the construction and analysis of higher-order FVMs was considered in [5] by Chen, Wu and Xu. For a specific quadratic scheme, its minimum angle condition can be obtained by a computer program, and the coercivity result is the same as that in [31] for the schemes in [10, 20, 24]. By choosing $(\alpha,\beta)=((3-\sqrt{3})/6,(3-\sqrt{3})/6)$, Zou [43] proposed an unconditionally stable quadratic scheme. Zhou and Wu [38,39] improved some coercivity results, e.g., the minimum angle should be greater than or equal to 10.08° for the scheme proposed in [24], 4.14° for the scheme proposed in [20], and 1.42° for the quadratic scheme proposed in [27] $(\alpha,\beta)=((3-\sqrt{3})/6,(6+\sqrt{3}-\sqrt{21+6\sqrt{3}})/9)$. Recently, [41] generalized the coercivity result in [38] to general anisotropic diffusion tensor, and also improved some existing minimum angle conditions. As a consequence, the coercivity analysis is one of the most challenging works for high order FVEMs.

Under the coercivity result of quadratic scheme over triangular meshes, one can study its convergence properties (e.g. H^1, L^2 and superconvergence) and apply it to solve more complicated problems (e.g. parabolic and nonlinear equations), see [4, 9, 14, 16, 26–28, 30, 33, 34, 37] for incomplete references. Some more studies of FVEM on triangular meshes can be found in [3, 5–7] and the references cited therein, and for the theoretical analysis of FVEM on quadrilateral meshes, we refer the reader to a non-exhaustive literature [15, 19, 22, 23, 32, 36, 40, 42].

We mention that the mapping from the trial function space to the test function space, plays an important role in the coercivity analysis of quadratic FVE schemes over triangular meshes. The general definition of this mapping is described in (3.1a) and (3.1b), depending on a parameter ω. The earlier and standard choice is $\omega=1$, e.g., [5,20,24,27,31]. Later, in order to construct an unconditionally stable quadratic scheme (resp. improve