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On a Right Inverse of a Polynomial of the Laplace in the Weighted Hilbert Space $L^2(\mathbb{R}^n, e^{-|x|^2})$

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Abstract. Let $P(\Delta)$ be a polynomial of the Laplace operator

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \quad \text{on} \quad \mathbb{R}^n.$$

We prove the existence of a bounded right inverse of the differential operator $P(\Delta)$ in the weighted Hilbert space with the Gaussian measure, i.e., $L^2(\mathbb{R}^n, e^{-|x|^2})$.

Key Words: Laplace operator, polynomial, right inverse, weighted Hilbert space, Gaussian measure.

AMS Subject Classifications: 35A01, 35A25, 35D30, 35J05

1 Introduction

In this paper, we study the right inverse of the polynomial differential operator of the Laplace

$$P(\Delta) = \Delta^m + a_{m-1}\Delta^{m-1} + \cdots + a_1\Delta + a_0,$$

where $a_0, a_1, \cdots, a_{m-1}$ are complex numbers. We prove the existence of global weak solutions of the equation $P(\Delta)u = f$ in the weighted Hilbert space $L^2(\mathbb{R}^n, e^{-|x|^2})$ by the following result of L^2 estimates.

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Theorem 1.1. *For each* $f \in L^2(\mathbb{R}^n, e^{-|x|^2})$, *there exists a weak solution* $u \in L^2(\mathbb{R}^n, e^{-|x|^2})$ *solving the equation*

$$P(\Delta)u = f$$

in $\mathbb{R}^n$ *with the norm estimate*

$$\int_{\mathbb{R}^n} |u|^2 e^{-|x|^2} dx \leq \frac{1}{(8n)^m} \int_{\mathbb{R}^n} |f|^2 e^{-|x|^2} dx.$$

The novelty of Theorem 1.1 is that the differential operator $P(\Delta)$ has a bounded right inverse

$$\begin{aligned} &Q : L^2(\mathbb{R}^n, e^{-|x|^2}) \quad \longrightarrow \quad L^2(\mathbb{R}^n, e^{-|x|^2}), \\ &P(\Delta)Q = I, \end{aligned}$$

with the norm estimate

$$\|Q\| \leq \frac{1}{(8n)^{\frac{m}{2}}}.$$

In particular, the Laplace operator Δ has a bounded right inverse

$$Q_0 : L^2(\mathbb{R}^n, e^{-|x|^2}) \quad \longrightarrow \quad L^2(\mathbb{R}^n, e^{-|x|^2}),$$

which, to the best of our knowledge, appears to be even new.

As a result, a natural question could be if Theorem 1.1 would be true for more general differential operators. For related results, see [1–4, 6]. The method employed in this paper was motivated from the Hörmander L^2 method [5] for Cauchy-Riemann equations in several complex variables.

The organization of this paper is as follows. In Section 2, we will prove several key lemmas based on functional analysis, while the proof of Theorem 1.1 will be given in Section 3. In Section 4, we will give some further remarks.

2 Several lemmas

In this section, we will prove the following lemma, which is key for the proof of Theorem 1.1.

Lemma 2.1. *Let* ξ *be a complex number. For each* $f \in L^2(\mathbb{R}^n, e^{-|x|^2})$, *there exists a weak solution* $u \in L^2(\mathbb{R}^n, e^{-|x|^2})$ *solving the equation*

$$\Delta u + \xi u = f \quad in \ \ \mathbb{R}^n,$$

with the norm estimate

$$\int_{\mathbb{R}^n} |u|^2 e^{-|x|^2} dx \leq \frac{1}{8n} \int_{\mathbb{R}^n} |f|^2 e^{-|x|^2} dx.$$