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Sharp Bound for the Generalized m-Linear n-Dimensional Hardy-Littlewood-Pólya Operator

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Abstract. In this paper, we calculate the sharp bound for the generalized m-linear n-dimensional Hardy-Littlewood-Pólya operator on power weighted central and noncentral homogeneous Morrey spaces. As an application, the sharp bound for the Hardy-Littlewood-Pólya operator on power weighted central and noncentral homogeneous Morrey spaces is obtained. Finally, we also find the sharp bound for the Hausdorff operator on power weighted central and noncentral homogeneous Morrey spaces, which generalizes the previous results.

Key Words: Sharp bound, n-dimensional Hardy-Littlewood-Pólya operator, power weight, Morrey space, Hausdorff operator.

AMS Subject Classifications: 42B25, 26D15

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1 Introduction

As a multilinear generalization of Calderón operator, the m-linear n-dimensional Hardy-Littlewood-Pólya operator is defined by

$$\mathcal{P}(f_1,\cdots,f_m)(x)=\int_{\mathbb{R}^{nm}}\frac{f_1(y_1)\cdots f_m(y_m)}{\max(|x|^n,|y_1|^n,\cdots,|y_n|^n)^m}dy_1\cdots dy_m. \tag{1.1}$$

Computation of the operator norm of integral operators is a challenging work in harmonic analysis. In 2006, Bényi and Oh [3] proved that for $n=1$,

$$\|P(f_1,\cdots,f_m)\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p}=\sum_{i=1}^m\prod_{j=1,\ j\neq i}^m p_j'.$$

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In fact, they proved sharp bound for certain multilinear integral operators that includes the Hardy-Littlewood-Pólya operator. In 2011, Wu and Fu [9] got the best estimate of the m-linear p-adic Hardy-Littlewood-Pólya operator on Lebesgue spaces with power weights. In 2017, Batbold and Sawano [2] studied one-dimensional m-linear Hilbert-type operators that includes Hardy-Littlewood-Pólya operator on weighted Morrey spaces, and they obtained the sharp bounds.

For the Hardy-Littlewood-Pólya operator over p-adic field, we refer to Fu et al. [5] and Li et. al. [6].

Inspired by [2,3,9], we will investigate a more general operator which includes the Hardy-Littlewood-Pólya operator as a special case and consider its operator norm on two power weighted Morrey spaces and its central version.

In the paper we use the following notation: For any measurable function w over a set E is given by

$$w(E)=\int_E wdx.$$

In what follows, $B(x,R)$ denotes the ball centered at x with radius R. Moreover, $|B(x,R)|$ denotes the Lebesgue measure of $B(x,R)$. Also, $B(0,R)$ denotes a ball of radius R centered at the origin.

We use this notation in the following definition of the weighted and weighted central homogeneous Morrey spaces.

Definition 1.1. *Let $w_1,w_2:\mathbb{R}^n\to(0,\infty)$ are positive measurable functions, $1\le q<\infty$ and $-1/q\le\lambda<0$. The weighted Morrey space $L^{q,\lambda}(\mathbb{R}^n,w_1,w_2)$ is defined by*

$$L^{q,\lambda}(\mathbb{R}^n,w_1,w_2)=\{f\in L^q_{loc}\,:\,\|f\|_{L^{q,\lambda}(\mathbb{R}^n,w_1,w_2)}<\infty\},$$

where

$$\|f\|_{L^{q,\lambda}(\mathbb{R}^n,w_1,w_2)}=\sup_{a\in\mathbb{R}^n,\ R>0} w_1(B(a,R))^{-(\lambda+1/q)}\left(\int_{B(a,R)}|f(x)|^q w_2(x)dx\right)^{1/q}.$$

Remark 1.1. When $w_1=w_2=1$, $L^{q,\lambda}(\mathbb{R}^n,w_1,w_2)$ is the classical Morrey spaces $L^{q,\lambda}(\mathbb{R}^n)$ and it was introduced by Morrey [8]. Note that $L^{q,-1/q}(\mathbb{R}^n)=L^q(\mathbb{R}^n)$, $L^{q,0}(\mathbb{R}^n)=L^\infty$ and $L^{q,\lambda}(\mathbb{R}^n)=\{0\}$ with $\lambda>0$. Based on the above reason, we only consider the case $-1/q<\lambda<0$.

Definition 1.2. *Let $w_1,w_2:\mathbb{R}^n\to(0,\infty)$ are positive measurable functions, $1\le q<\infty$ and $-1/q\le\lambda<0$. The weighted central homogeneous Morrey space $\dot{M}^{q,\lambda}(\mathbb{R}^n,w_1,w_2)$ is defined by*

$$\dot{M}^{q,\lambda}(\mathbb{R}^n,w_1,w_2)=\{f\in L^q_{loc}\,:\,\|f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,w_1,w_2)}<\infty\},$$

where

$$\|f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,w_1,w_2)}=\sup_{R>0} w_1(B(0,R))^{-(\lambda+1/q)}\left(\int_{B(0,R)}|f(x)|^q w_2(x)dx\right)^{1/q}. \tag{1.2}$$