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Neural Network Method for Integral Fractional Laplace Equations

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Abstract. A neural network method for fractional order diffusion equations with integral fractional Laplacian is studied. We employ the Ritz formulation for the corresponding fractional equation and then derive an approximate solution of an optimization problem in the function class of neural network sets. Connecting the neural network sets with weighted Sobolev spaces, we prove the convergence and establish error estimates of the neural network method in the energy norm. To verify the theoretical results, we carry out numerical experiments and report their outcome.

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Key words: Deep Ritz method, neural network, fractional elliptic PDE, ReLU.

1. Introduction

Fractional Laplacian operator [8, 14, 34] has been employed in various nonlocal models, including turbulence [6, 21, 22], quantum mechanics [26, 33], finances [12], statistical physics [23], phase transitions [4, 5], material sciences [7], image processing [24], geophysics [13], acoustic wave propagation in heterogeneous media [45] and anomalous diffusion in porous media [35, 36]. Due to versatile applications and the ability to capture anomalous diffusion and model complex physical phenomena with long range interaction [16,39,42], many numerical methods have been developed — e.g. finite difference-quadrature methods [18,19,30,32,37], finite element methods (FEM) [1,3,15,41], spectral methods [28,29], mesh-free pseudospectral methods [10,40], the isogeometric collocation method [43], and deep learning method [38]. We refer the readers to [8, 14, 34] for a review of many definitions of fractional Laplacian and their numerical methods.

In this paper, we consider an n-dimensional fractional diffusion equation with Dirichlet boundary condition — viz.

$$(-\Delta)^{\frac{\alpha}{2}}u = f(x), \quad x\in\Omega, \quad \alpha\in(0,2), \tag{1.1}$$

$$u(x) = 0, \qquad x\in\Omega^c, \tag{1.2}$$

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where Ω and Ω^c are a domain and its complement in $\mathbb{R}^n$, $n = 1,2,3$, $f(x)$ is a given function, and the fractional Laplacian is defined by

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+\alpha}} dy, \quad c_{n,\alpha} = \frac{2^\alpha \Gamma((\alpha+n)/2)}{\pi^{\frac{n}{2}} |\Gamma(-\alpha/2)|}. \tag{1.3}$$

The essential difficulties for the fractional diffusion equations are twofold:

(1) The most of the numerical methods mentioned exhibit a low convergence order and accuracy because the solution has a singularity near the boundary inherited from the kernel.

(2) For $\alpha \in (1,2)$, the complexity caused by the singular integral makes the corresponding discretization challenging.

For special domains such as a disk in two-dimensional space or a ball in three-dimensional space, one can construct very accurate high-order methods based on pseudo-eigenfunctions of the fractional Laplacian operator [28]. For general domains any other higher order methods are not known. The FEM proposed in [1] on a graded mesh can capture the singularity near the boundary and recover the optimal convergence order for the piecewise linear polynomials. Nonetheless, the assembling the stiffness matrix is expensive, and it is still under investigation for high-order polynomials. Therefore, here we turn our attention to the current state-of-the-art neural network methods (NNMs). Neural network methods have gained increased attention recently in the science and engineering [20, 44]. The methods are powerful in approximation and have huge expressive power. Although many numerical methods use meshes, which are often constructed in prior or posterior in an adaptive way, they can be understood and classified into mesh-free methods. This make it easy to capture boundary or corner singularities and recover boundary and transition layers and the shocks encountered in hyperbolic problems. Their success for the integer-order problems [20, 44] naturally motives us to apply the methods to the model problem (1.1)-(1.2).

Compared to extensive research on integer-order local problems, there are limited research on neural networks for fractional counterparts incorporating nonlocalities. Pang *et al.* [38] applied the so-called FPINN method to a advection-diffusion equation with the fractional Laplacian. They showed numerical convergence of the method without theoretical analysis. Combining with the Monte Carlo numerical integration, Guo *et al.* [27] extended the FPINN method to high-dimensional forward and inverse problems. We remark that although their least-squares formulation works well for $\alpha \in (0,1)$, it experiences problems if $\alpha \in (1,2)$. Since the solution does not have enough regularity to use the least-squares form, numerical solution for $\alpha \in (1,2)$ was not reported in [38]. To resolve this issue we use the energy formulation of the model problem (1.1)-(1.2). Note that it is called the deep Ritz method in [20].

The main goal of this paper is the development of a more accurate neural network method for the model problem. The neural network method is set up for the energy form