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Extremum-Preserving Correction of the Nine-Point Scheme for Diffusion Equation on Distorted Meshes

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Abstract. In this paper, we construct a new cell-centered nonlinear finite volume scheme that preserves the extremum principle for heterogeneous anisotropic diffusion equation on distorted meshes. We introduce a new nonlinear approach to construct the conservative flux, that is, a linear second order flux is firstly given and a nonlinear conservative flux is then constructed by using an adaptive method and a nonlinear weighted method. Our new scheme does not need to use the convex combination of the cell-center unknowns to approximate the auxiliary unknowns, so it can deal with the problem with general discontinuous coefficients. Numerical results show that our new scheme performs more robust than some existing schemes on highly distorted meshes.

AMS subject classifications: 65N08, 65M08

Key words: Extremum-preserving correction, diffusion equation, distorted mesh, nine-point scheme.

1 Introduction

The extremum principle is one of important features of diffusion equation. But, it is difficult to construct a numerical scheme that preserves the extremum principle on distorted

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meshes for general diffusion equation. As we know, the numerical schemes which preserve the monotonicity–a special case of the minimum principle, have been discussed in [7, 13, 16–18, 23, 31]. However, these monotone schemes may not satisfy the maximum principle for the general case, because these monotone schemes can maintain either a lower bound or an upper bound, but not both at the same time (see the numerical examples in Subsection 4.2).

Some cell-centered finite volume schemes that preserve the extremum principle have been presented in [4, 8, 10–12, 25, 27, 29, 30, 32]. These schemes use certain nonlinear methods in constructing the conservative flux. A finite volume scheme satisfying the extremum principle for the convection-diffusion is given in [4]. Discrete maximum principle for Galerkin approximations of the Laplace operator on arbitrary meshes is considered in [6]. Some theoretical analysis are given for the scheme satisfying the extremum principle under some coercivity assumption in [11]. A scheme preserving the extremum principle for the diffusion equation on star-shaped polygonal meshes is presented in [25], which is based on nonlinear weighted method. A general construction of finite volume scheme preserving the extremum principle is given in [27].

In the construction of finite volume schemes [27] that preserve the extremum principle for diffusion equations on distorted meshes, certain auxiliary unknowns, such as edge-midpoint unknows or vertex unknowns, are usually introduced in order to exactly deal with possible discontinuity of diffusion coefficients. To obtain a finite volume scheme with only cell-centered unknowns, these auxiliary unknowns must be eliminated by expressing them as a combination of neighboring cell-centered unknowns. But this combination is not convex one in general. Anyway, the nonnegativity of these coefficients and their sum being one are necessary to guarantee that any resulting cell-centered finite volume scheme satisfies the extremum principle.

For highly distorted meshes or the case there are multiple discontinuous lines for the diffusion coefficient, it is difficult to find the high-order convex combination approximation for the auxiliary unknowns. As we know that most existing cell-centered finite volume schemes can satisfy the extremum principle only in some special cases, such as the case that there is only one discontinuous line in the whole domain [25, 27]. In addition, on highly distorted meshes, the accuracy of these extremum-preserving finite volume schemes is lower than that of the linear finite volume schemes for solving diffusion equations, such as, the nine-point scheme [24] or the multi-point flux method (MPFA) [1].

In order to avoid the difficulty of approximating auxiliary variables by locally convex interpolation, some interpolation-free methods have been proposed to avoid the interpolation procedure [9, 18, 22, 23], but they have some restrictions on the location of cell-centers or coefficient discontinuity. Besides, a nonlinear technique to correct the linear finite volume schemes for anisotropic diffusion problems is presented in [8], which removes the restrictions on cell-centers selection but has lower accuracy. An approach based on algebraic flux correction for constructing a nonlinear extremum-preserving finite volume scheme is given in [29, 30], which has second-order numerical precision.

In this paper, we use an idea similar to [29, 30] by modifying the discrete flux of the