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# Nonconforming Mixed FEM Analysis for Multi-Term Time-Fractional Mixed Sub-Diffusion and Diffusion-Wave Equation with Time-Space Coupled Derivative

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**Abstract.** The main contents of this paper are to establish a finite element fully-discrete approximate scheme for multi-term time-fractional mixed sub-diffusion and diffusion-wave equation with spatial variable coefficient, which contains a time-space coupled derivative. The nonconforming $EQ_1^{rot}$ element and Raviart-Thomas element are employed for spatial discretization, and L1 time-stepping method combined with the Crank-Nicolson scheme are applied for temporal discretization. Firstly, based on some significant lemmas, the unconditional stability analysis of the fully-discrete scheme is acquired. With the assistance of the interpolation operator $I_h$ and projection operator $R_h$, superclose and convergence results of the variable $u$ in $H^1$-norm and the flux $\vec{p}=\kappa_5(\mathbf{x})\nabla u(\mathbf{x},t)$ in $L^2$-norm are obtained, respectively. Furthermore, the global superconvergence results are derived by applying the interpolation postprocessing technique. Finally, the availability and accuracy of the theoretical analysis are corroborated by experimental results of numerical examples on anisotropic meshes.

**AMS subject classifications**: 65M12, 65N30, 65N15

**Key words**: Multi-term time-fractional mixed sub-diffusion and diffusion-wave equation, nonconforming $EQ_1^{rot}$ mixed FEM, L1 approximation and Crank-Nicolson scheme, convergence and superconvergence.

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# 1 Introduction

For adhibitions of fractional partial differential equations(FPDEs) involve in a wide range, which are becoming more and more popular. The FPDEs are major for simulating and analysis, especially in terms of heredity and variation, the FPDEs have more advantages than integer ones. Therefore, in many industries or subjects, they also play a decisive role. For instance, [1] presented the chaotic dynamics of classical conservative systems. Detailed studies on groundwater contaminant transport had been done in [2]. [3] illustrated fractional order diffusion equations which were chosen in this study for modeling the anomalous mass transport in the rocks. [4] talked over the expression and properties of Green's function for boundary value problems of nonlinear Sturm-Liouville-type fractional order impulsive differential equations. Based on a generalized fuzzy hyperbolic model, [5] put forward a new approach which was used for the numerical solution of variable order fractional differential algebraic equations. The behavioral responses of the lumped capacitance model with tempered fractional order for transient conduction were investigated in [6]. [7] showed the Caputo-Fabrizio fractional derivative and the Laplace transform technique which applied to develop exact analytical solutions for velocity and temperature profiles. [8] clarified the application of the fractional differential equations in modeling the respiratory syncytial virus infection. A numerical scheme and an alternating direction implicit scheme for time-space fractional vibration equations were established in [9], and the convergence and unconditional stability of the proposed schemes were proved rigorously. The backward Euler and Crank-Nicolson-Galerkin fully-discrete approximate schemes for two-dimensional space-fractional advection-dispersion equations were established in [10]. Applying finite element method in spatial direction and classical L1 approximation in temporal direction in [11], a fully-discrete scheme was established for a class of two-dimensional multi-term time fractional diffusion equations with Caputo fractional derivatives.

In addition, aiming at multi-term time fractional diffusion equations and multi-term time fractional diffusion wave equations, many experts and scholars have penetrated into discussing and studied their properties and applications from various aspects, here are a great quantity of examples as regards these two kinds of equations. [12–14] proposed a fully discrete local discontinuous Galerkin method, a numerical scheme and the general exact solutions, respectively. Two alternating direction implicit difference schemes, some computationally effective numerical methods and a Legendre spectral tau method were derived in [15–17], respectively. [18] aimed to identify a time-dependent source term in a multi-term time-fractional diffusion equation from the boundary Cauchy data. [19] discussed a meshless collocation method which was considered to solve the multi-term time fractional diffusion-wave equation in two dimension. [20] studied an implicit numerical scheme for a class of multi-term time-fractional diffusion equation. [21] considered a multi-term variable-order fractional diffusion equation on a finite domain, which involved the Caputo variable-order time fractional derivative of order $\alpha(\mathbf{x},t)\in(0,1)$ and the Riesz variable-order space fractional derivatives of order $\beta(\mathbf{x},t)\in(0,1)$, $\gamma(\mathbf{x},t)\in(1,2)$.