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Achieving Superconvergence by One-Dimensional Discontinuous Finite Elements: The CDG Method

Xiu Ye[1] and Shangyou Zhang[2,*]

[1] *Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA.*
[2] *Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA.*

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Abstract. Novelty of this work is the development of a finite element method using discontinuous P_k element, which has two-order higher convergence rate than the optimal order. The method is used to solve a one-dimensional second order elliptic problem. A totally new approach is developed for error analysis. Superconvergence of order two for the CDG finite element solution is obtained. The P_k solution is lifted to an optimal order P_{k+2} solution elementwise. The numerical results confirm the theory.

AMS subject classifications: 65N15, 65N30

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1. Introduction

In order to approximate the solutions of partial differential equations, the conforming discontinuous Galerkin method [5] and the interior penalty discontinuous Galerkin (IPDG) method [1] use a discontinuous P_k polynomial. On the other hand, their continuous counterpart conforming finite element method employs a continuous P_k element. Since the discontinuous P_k polynomial allows many more degrees of freedom, one could expect a higher order convergence rate for discontinuous finite element method. However, both IPDG and CDG have the same optimal convergence rates as their continuous counterpart. Can we develop a finite element method with a discontinuous P_k element, which would utilize all additional unknowns introduced by the discontinuity of P_k and achieve a higher order convergence rate than its continuous counterpart? This is the question we want to answer in this paper.

The CDG method obtains its name by combining the features of conforming finite element and discontinuous Galerkin (DG) finite element methods. It has the flexibility of discontinuous approximations and the simplicity of the conforming finite elements. This

*Corresponding author. *Email addresses:* `xxye@ualr.edu` (X. Ye), `szhang@udel.edu` (S. Zhang)

method shares the same finite element space as the IPDG method. The reason while the CDG method preserves its simple formulation consists in using weak derivatives as in the weak Galerkin (WG) method. Weak Galerkin finite element methods, introduced in [3] exploit general finite element technique for solving partial differential equations. The novelty of such methods consists in the introduction of a weak function and its weakly defined derivative. Weak functions have the form $v = \{v_0, v_b\}$, where $v = v_0$ and $v = v_b$ are the values of v in the interior of each element and on the boundary of the element, respectively. The modified weak Galerkin (MWG) method in [4] and the CDG method can be derived by eliminating v_b in the WG method and replacing it by the average of v_0. The CDG method can be viewed as a stabilizer free MWG method. The CDG methods have been developed for diffusion problem [2,5,6], and for biharmonic equations [7].

The purpose of this work is the development of a finite element method with discontinuous P_k element with order two superconvergence rate. This is a challenging project, and here we start with the one dimensional problem

$$\begin{aligned} -u'' &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega = [a, b]$.

We develop a new CDG method which fully utilizes all degrees of freedom of the discontinuous P_k polynomial to achieve two-order higher convergence rate than its counterpart conforming finite element method using continuous P_k element. This paper is the continuation of the work [8], where we proved that the corresponding WG method has order two superconvergence rate. It is much more difficult to develop a CDG method with order two superconvergence because it has many fewer unknowns (without unknown v_b). In this new CDG method, v_b is calculated by a new sophisticated scheme and a novel approach to error analysis is created to deal with this difficult problem. Order two superconvergence of the CDG solution is proved theoretically and confirmed numerically. We also lift such a P_k CDG solution to an optimal order P_{k+2} solution elementwise. Finally, in 1-D situation we answer the question whether a well defined finite element method with discontinuous P_k element can converge two-order faster than the finite element method with continuous P_k element.

2. CDG Finite Element Method

Let $\Omega = [a, b] = \cup_{i=1}^{N} I_i$ with $I_i = [x_{i-1}, x_i]$, $\mathcal{T}_h := \{I_i \mid i = 1, \ldots, N\}$, $h = \max |I_i|$, and $Dv := dv/dx$. The conforming discontinuous Galerkin finite element space is defined by

$$V_h := \left\{ v \in L^2(\Omega),\ v|_I \in P_k(I),\ I \in \mathcal{T}_h \right\}. \tag{2.1}$$

In order to describe the CDG method, we consider a weak Galerkin finite element space — viz.

$$\tilde{V}_h := \left\{ v = \{v_0, v_b\} : v_0|_I \in P_k(I),\ v_b|_x \in P_0(x),\ x \subset \partial I, I \in \mathcal{T}_h, v_b|_{\partial\Omega} = 0 \right\}.$$