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Numerical Studies of Vanka-Type Smoothers in Computational Solid Mechanics

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Abstract. In this paper multigrid smoothers of Vanka-type are studied in the context of Computational Solid Mechanics (CSM). These smoothers were originally developed to solve saddle-point systems arising in the field of Computational Fluid Dynamics (CFD), particularly for incompressible flow problems. When treating (nearly) incompressible solids, similar equation systems arise so that it is reasonable to adopt the 'Vanka idea' for CSM. While there exist numerous studies about Vanka smoothers in the CFD literature, only few publications describe applications to solid mechanical problems. With this paper we want to contribute to close this gap. We depict and compare four different Vanka-like smoothers, two of them are oriented towards the stabilised equal-order Q_1/Q_1 finite element pair. By means of different test configurations we assess how far the smoothers are able to handle the numerical difficulties that arise for nearly incompressible material and anisotropic meshes. On the one hand, we show that the efficiency of *all* Vanka-smoothers heavily depends on the proper parameter choice. On the other hand, we demonstrate that only *some* of them are able to robustly deal with more critical situations. Furthermore, we illustrate how the enclosure of the multigrid scheme by an outer Krylov space method influences the overall solver performance, and we extend all our examinations to the nonlinear finite deformation case.

AMS subject classifications: 65N22, 65N55, 65H10, 65F50, 74S05

Key words: Coupled multigrid, Vanka smoother, linear and finite elasticity, nearly incompressible material, saddle point systems, finite elements.

1 Introduction

Multigrid solvers rank among the most efficient solvers in many application fields. They are especially suited for solving large linear equation systems stemming from

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discretisations of elliptic partial differential equations. In this paper we deal with such systems arising in the context of Computational Solid Mechanics (CSM). The continuous problems are discretised using the Finite Element method.

The efficiency and the robustness of the (geometric) multigrid method crucially depends on the smoothing operator. We want to examine a class of smoothers which was originally introduced by Vanka [35] for solving the Navier-Stokes equations discretised by Finite Differences. Basically, the method can be described as a block Gauß-Seidel iteration, which locally couples all field variables occuring in the formulation. The smoother is sometimes denoted as *symmetrically coupled Gauß-Seidel (SCGS)* or *box iteration/relaxation*. Compared to standard (point-wise) Jacobi or Gauß-Seidel smoothers, the crucial advantage of the Vanka approach is the ability to deal with zero blocks appearing on the diagonal of the system matrix. Saddle point systems stemming from discretisations of the incompressible Navier-Stokes equations have this property which is the main reason for the strong influence the method had (and still has) in the field of Computational Fluid Dynamics (CFD). Other reasons are that it is not too difficult to implement and at the same time efficient and robust for a wide class of problem configurations.

The Vanka approach has to be seen in contrast to the class of multigrid smoothers which treat the system in a global (and eventually decoupled) manner [8,24,32]. For a comparison between the different approaches in the context of CFD see, for example, the contributions of John, Tobiska [12,15] and Turek [32]. For further references, see the overview paper of Wesseling and Oosterlee [36].

While there seem to be only few papers dealing with theoretical aspects of the smoother [20,21,29], much literature can be found presenting numerical studies of different Vanka-type smoothers for solving the discretised Navier-Stokes equations in CFD. John and Tobiska apply it to the non-conforming Crouzeix/Raviart element P_1/P_0, Turek to the corresponding non-conforming rotated bilinear Rannacher/Turek element $\tilde{Q}_1/P_0$ and Becker to the stabilised Q_1/Q_1 element [3,12,15,32]. In all cases, the smoother is extensively tested on the benchmark configuration 'Flow around a cylinder' [33] for the steady and unsteady state. Ouazzi and Turek [22] transfer the Vanka idea to edge-oriented storage- and stabilisation techniques for Navier-Stokes equations. Zeng and Wesseling [38] compare Vanka-type smoothers to ILU methods for the case of Navier-Stokes in general coordinates. To treat anisotropic grids more robustly, several extensions have been introduced. Thompson and Ferziger define *symmetrically coupled alternating line (SCAL)* versions for Finite Difference discretisations, Becker uses a *string-wise* version for the stabilised Q_1/Q_1 discretisation, and Schmachtel develops an adaptive blocking strategy [3,28,31]. John and Matthies successfully apply Vanka smoothers to higher order finite element methods [13,14]. Comparative solver studies including Vanka smoothers can be found in the articles of Benzi and Olshanskii [4] and Larin and Reusken [18].

There are only few papers describing the use of Vanka-type smoothers in the context of CSM. For many kinds of solid mechanical problems there is obviously no need to refrain from standard (point-wise) multigrid smoothers. But for special formula-