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Waves Induced by the Appearance of Single-Point Heat Source in Constant Flow

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Abstract. Heating or cooling one-dimensional inviscid compressible flow can be modeled by the Euler equations with energy sources. A tricky situation is the sudden appearance of a single-point energy source term. This source is discontinuous in both the time and space directions, and results in multiple discontinuous waves in the solution. We establish a mathematical model of the generalized Riemann problem of the Euler equations with source term. Based on the double CRPs coupling method proposed by the authors, we determine the wave patterns of the solution. Theoretically, we prove the existence and uniqueness of solutions to both "heat removal" problem and "heat addition" problem. Our results provide a theoretical explanation for the effect of instantaneous addition or removal of heat on the fluid.

AMS subject classifications: 35L81, 80A20

Key words: Hyperbolic balance law, generalized Riemann problem, singular source, existence and uniqueness.

1 Introduction

In physical problems involving fluids, the addition or removal of mass and energy is mathematically expressed in the form of source terms. A special case is that the source term is discontinuous in time or space. For example, for a three-dimensional example of the evaporation of a spherical water drop in the air, the effect of evaporation on air is a process of addition of mass (the diffusion of water vapor into air) and absorption of energy. Note that such a process of evaporation only occurs on the interface between the water droplet and the air, thus the existence of the source term is only a two-dimensional surface, which has a dimension less than the space where the problem under consideration lies. Such a special source term can be mathematically expressed in the Dirac delta function, and its one-dimensional representation is

$$\delta(x)=\frac{dH}{dx}, \quad \text{where } H(x)=\begin{cases} 0, & x\leq 0, \\ 1, & x>0. \end{cases}$$

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This function is discontinuous in space. On the other hand, the appearance and disappearance of sources is also discontinuous in time. This type of source that is discontinuous in time and space generally induces the generation of multiple waves. The properties of these waves are important in both theoretical research and algorithm design, and this is the reason that prompted us to conduct the research in this paper.

In this paper we only consider a simple one-dimensional model

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0, \tag{1.1a}$$

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2+p)}{\partial x}=0, \tag{1.1b}$$

$$\frac{\partial E}{\partial t}+\frac{\partial(Eu+pu)}{\partial x}=Q\delta(x)H(t). \tag{1.1c}$$

The thermodynamical variables ρ, E, p are density, total energy and pressure, respectively. u is velocity. The first two equations in (1.1) describe the law of conservation of mass and momentum, respectively. The Q in the energy equation (the third equation in (1.1)) is related to the state of fluid, and the $\delta(x)$ means that the source is only distributed on a single point $x=0$, and $H(t)$ means that the source term starts to appear at time $t=0+$. The energy equation describes the heat addition or removal to the fluid. If $Q=0$, the energy of the flow is conserved and (1.1) is the classical Euler equations. If $Q>0$, $|Q|$ heat is added to the flow per unit time, which is called heat addition problem. Conversely, if $Q<0$, $|Q|$ heat is removed from the fluid per unit time, which is called heat removal problem. We assume that the flow is steady before the source appears, which implies

$$U(x,t)\equiv U_0=\begin{pmatrix}\rho_0\\ \rho_0 u_0\\ E_0\end{pmatrix} \quad \text{for} \ \ t\leq 0.$$

The solution of (1.1) is a singularity decomposition process of the source term at the origin, hence we consider a generalized Riemann problem, which is

$$\begin{cases} \dfrac{\partial U}{\partial t}+\dfrac{\partial F}{\partial x}=S,\\ U(x,0)\equiv U_0, \end{cases} \tag{1.2}$$

where

$$U=\begin{pmatrix}\rho\\ \rho u\\ E\end{pmatrix}, \quad F=\begin{pmatrix}\rho u\\ \rho u^2+p\\ (E+p)u\end{pmatrix}, \quad S=\begin{pmatrix}0\\ 0\\ Q\delta(x)\end{pmatrix}.$$

U is the vector of conserved variable, F is the vector of flux function. Note that the $H(t)$ in the energy equation of (1.1) is implicitly included in the initial condition. An equation of state is required to complete the system. We will restrict ourselves to the ideal gases, which satisfies

$$p=(\gamma-1)\rho e, \tag{1.3}$$