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A Weak Galerkin Mixed Finite Element Method for Acoustic Wave Equation

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Abstract. This paper is concerned with the weak Galerkin mixed finite element method (WG-MFEM) for the second-order hyperbolic acoustic wave equation in velocity-pressure formulation. In this formulation, the original second-order differential equation in time and space is reduced to first-order differential equations by introducing the velocity and pressure variables. We employ the usual discontinuous piecewise-polynomials of degree $k \geq 0$ for the pressure and $k+1$ for the velocity. Furthermore, the normal component of the pressure on the interface of elements is enhanced by polynomials of degree $k+1$. The time derivative is approximated by the backward Euler difference. We show the stability of the semi-discrete and fully-discrete schemes, and obtain the suboptimal order error estimates for the velocity and pressure variables. Numerical experiment confirms our theoretical analysis.

AMS subject classifications: 65M60, 65M12

Key words: Acoustic wave equation, velocity-pressure formulation, WG-MFEM.

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1 Introduction

The propagation phenomena of waves (e.g., acoustic, electromagnetic or elastic waves), which are governed by a class of second-order hyperbolic equations, were extensively investigated because their varied media are often anisotropic heterogeneous or have time-dependent properties, and the solutions of the hyperbolic conservation laws often contain discontinuities. Thus, some efficient numerical methods have been employed on the hyperbolic models to handle the problems, for instance, the Galerkin methods [9], the mixed finite element methods (mixed FEMs) [4,13], the discontinuous Galerkin (DG) methods [2,10], the stabilized FEMs [3], the multiscale methods [1] and so on. In addition, the geometry complexity of media such as cracks, obstacles and arbitrarily-shaped boundaries, also results in difficulties, and therefore, other effective methods with advantages in mesh generation are developed to conquer this problem.

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In recent years, the FEMs which are capable of dealing with the polygonal and polyhedral meshes have caused widespread concern, and an incomplete list is: the hybrid discontinuous Galerkin (HDG) methods [7], the local discontinuous Galerkin (LDG) methods [33], the hybrid high-order (HHO) methods [8] and the virtual element methods (VEMs) [24,31].

The weak Galerkin (WG) method introduced in [30] for the second-order elliptic equations, also plays an important roll in treating polygonal or polyhedral meshes. The method provides a new competitive numerical technique for the FEM by introducing the weakly defined differential operators (gradient, divergence, curl and Laplacian) to replace classical ones. The WG method extends the traditional finite element partitions of triangles or quadrilaterals in 2D and tetrahedrons or hexahedrons in 3D to these of general polygons and polyhedrons by adding some parameter-free stabilization terms which enforce the weak continuity [19]. Specifically, the WG method approximates unknowns with discontinuous polynomials on general meshes and does not have to choose the penalty parameter carefully. On the other hand, the WG method is promoted actively with some pioneer works. A computational investigation about numerical interpolations of the WG method was conducted in [17]. Minimizing unknowns without compromising the accuracy of the numerical approximation was realized in [22] by optimal combinations of the polynomial spaces. Optimal order error estimates of the WG-MFEM for the second-order elliptic equations have been established in [26]. Furthermore, due to its significant flexibility in mesh generation, the WG method has been applied successfully to a wide range of problems such as the Stokes equations [27], the Oseen equations [15], the Navier-Stokes equations [11, 16], the Biharmonic equation [20, 25], the Maxwell equations [23], the Brinkman equations [29,32], the Helmholtz equation [21,28], the convection-diffusion-reaction equations [5], the quasi-Newtonian Stokes flows [34], the elliptic interface problems [18], and the coupling Stokes-Darcy problem [6,14].

Additionally, [12] applied the WG method to the second-order hyperbolic acoustic wave equation in the original displacement formulation, and analysed the corresponding stability and convergence. For the reason that WG-MFEM inherits the advantages of the WG method and can give a direct finite element approximation of the flux variable, utilizing the WG-MFEM to the wave equations is of great significance. In this paper we introduce a velocity variable and a flux (pressure) variable to reduce the original second-order acoustic wave equation to first-order ones in time and space respectively, and establish a WG mixed finite element scheme based on the velocity-pressure formulation. For the numerical approximation, the velocity variable is approximated by using discontinuous piecewise-polynomials of degree $k+1$ while the pressure variable is approached by utilizing discontinuous piecewise-polynomials of degree k. Furthermore, the normal component of pressure is enhanced by the polynomials of degree $k+1$ on the edges or faces of elements. We prove the existence and uniqueness of the numerical solution for the fully-discrete schemes. We also analyze the stability and obtain the suboptimal order error estimates for the semi-discrete and fully-discrete schemes. Finally, our theoretical analysis is validated by the numerical experiment.