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Two-Grid Immersed Finite Volume Element Methods for Semi-Linear Elliptic Interface Problems with Non-Homogeneous Jump Conditions

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Abstract. In this paper, we propose an immersed finite volume element method for solving the semi-linear elliptic interface problems with non-homogeneous jump conditions. Furthermore, two-grid techniques are used to improve the computational efficiency. In this way, we only need to solve a non-linear system on the coarse grid, and a linear system on the fine grid. Numerical results illustrate that the proposed method can solve the semi-linear elliptic interface problems efficiently. Approximate second-order accuracy for the solution in the L^∞ norm can be obtained for the considered examples.

AMS subject classifications: 65N30

Key words: Two-grid, immersed finite volume element, Cartesian mesh, semi-linear, non-homogeneous.

1 Introduction

Let $\Omega \in R^2$ be a bounded convex polygonal domain with boundary $\partial\Omega$. We consider the following semi-linear elliptic interface problem

$$\begin{cases} -\nabla\cdot(\beta(x)\nabla u)+d(u)=f(x) & \text{in } \Omega^{\pm}, \\ u|_{\partial\Omega}=g(x), \end{cases} \tag{1.1}$$

where the domain Ω is assumed to be composed of two sub-domains Ω^+ and Ω^- divided by an interface Γ, $d(u)$ is a nonlinear function, $f(x)$ and $g(x)$ are given real-valued

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functions. The coefficient $\beta(x)$ has a finite discontinuity across the interface Γ, and it is assumed to be a piecewise constant function defined by

$$\beta(x)=\begin{cases} \beta^-, & x=(x_1,x_2)\in\Omega^-, \\ \beta^+, & x=(x_1,x_2)\in\Omega^+, \end{cases} \tag{1.2}$$

such that $\min\{\beta^+,\beta^-\}>0$. Across the interface Γ, the solution and its gradient in the normal direction satisfy the following jump conditions

$$[u]|_\Gamma=u^+-u^-=J_1(x), \quad \left[\beta\frac{\partial u}{\partial \mathrm{n}}\right]\bigg|_\Gamma=\beta^+\frac{\partial u^+}{\partial \mathrm{n}}-\beta^-\frac{\partial u^-}{\partial \mathrm{n}}=J_2(x), \tag{1.3}$$

where $u^s=u|_{\Omega^s}$, $s=+$ or $-$, n is the unit outward normal vector of Γ, $J_1(x)$ and $J_2(x)$ are given real-valued functions on Γ. The interface Γ is represented by the zero level-set of a function $\varphi(x)$ which is called a level-set function, and we assume that $\Omega^-=\{x\in\Omega|\varphi(x)<0\}$ and $\Omega^+=\{x\in\Omega|\varphi(x)>0\}$.

The interface problem (1.1)-(1.3) has many applications in fluid dynamics [6,32], material science [25] and biological science [44,60], etc. A large number of numerical methods are proposed in the literature to solve the interface problems with $d(u)=0$. Classical finite element methods can solve the elliptic interface problems satisfactorily if the triangulation is aligned with the interface [3,5,13]. However, it takes additional cost to reform the mesh at each time step for moving interface problems. Since Peskin's pioneering work of the immersed boundary method [45], many numerical methods have been constructed based on finite difference discretization on Cartesian mesh [17,31,33,35,46,61]. To avoid the complicated mesh generation process, the extended finite element methods have been proposed to allow the interface to cut through elements so that Cartesian mesh can be employed [43,47,58].

The immersed finite element (IFE) method was also developed for solving interface problems efficiently on structured meshes independent of the interface [29, 36, 39, 41]. The key idea of IFE method is to use piecewise polynomials constructed according to the jump conditions on interface elements, while standard polynomials are used on other elements. The IFE method results in a symmetric positive definite linear system when the underlying problems are symmetric positive, and second-order accuracy in the L^2 norm can be obtained. To solve the elliptic interface problem with non-homogeneous jump conditions, the homogenization technique based on the level-set idea is proposed, and an equivalent elliptic interface problem with homogeneous jump conditions is formulated [16]. In [8, 23], a new IFE function is constructed on each interface element to approximate the non-homogeneous jump conditions. However, these IFE method does not have optimal convergence in L^∞ norm, and has a much larger pointwise error over interface. Inspired by the discontinuous Galerkin method [2, 57], the partially penalized IFE (PPIFE) method are proposed in some references [19, 27, 28, 30, 40, 53]. In [26], a Petrov-Galerkin finite element formulation with non-body-fitting grid is proposed to solve elliptic interface problems. In their approach, non-homogeneous jump conditions