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Sobolev Orthogonal Polynomials: Asymptotics and Symbolic Computation

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Abstract. The Sobolev polynomials, which are orthogonal with respect to an inner product involving derivatives, are considered. The theory about these nonstandard polynomials has been developed along the last 40 years. The local asymptotics of these polynomials can be described by the Mehler-Heine formulae, which connect the polynomials with the Bessel functions of the first kind. In recent years, the formulae have been computed for discrete Sobolev orthogonal polynomials in several particular cases. We improve various known results by unifying them. Besides, an algorithm to compute these formulae effectively is presented. The algorithm allows to construct a computer program based on *Mathematica*® language, where the corresponding Mehler-Heine formulae are automatically obtained. Applications and examples show the efficiency of the approach developed.

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1. Introduction

Sobolev orthogonal polynomials were first considered in a paper concerning the simultaneous polynomial approximation of functions and their derivatives in 1947 [17]. In the sixties and seventies of the last century, the German school studied such polynomials, mainly from a theoretical point of view [4, 11, 26, 27]. At the beginning of the nineties, the topic attracted great interest after the seminal works of Iserles *et al.* [12, 13], where among other results the authors proposed a useful algorithm for computing the coefficients of the Fourier-Sobolev series. More details about the Sobolev orthogonality can be found mainly in the survey [21], but a very short and concise overview is given in the first two pages of [20]. More recently, several authors have found applications of this theory — e.g. in the study of second-order elliptic equations on the real line [25], in elliptic boundary

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value problems combining Jacobi-Sobolev polynomials and spectral methods [31], and in the Cauchy problem for ordinary differential equations [28,29].

In this paper we consider the Sobolev-type inner product or the discrete Sobolev inner product

$$(f,g)_S = \int f(x)g(x)d\mu + M_n f^{(j)}(c)g^{(j)}(c), \tag{1.1}$$

where μ is a finite positive Borel measure supported on an infinite subset of the real line, c is adequately located on the real axis, $j \in \mathbb{N}\cup\{0\}$, and $\{M_n\}_{n\geq 0}$ is a sequence of nonnegative real numbers satisfying a very general condition.

The orthonormal polynomials with respect to (1.1) are studied from a theoretical point of view in [19]. There the authors provided a scaled asymptotics around the point c where the perturbation of the standard inner product is introduced. This type of local asymptotics is known either as Mehler-Heine formula or Mehler-Heine asymptotics. As it is well known, it is quite useful to describe the asymptotic differences between the sequences of orthogonal polynomials with respect to (1.1) and the orthonormal polynomials with respect to μ. In fact, in that paper it was shown that the relative asymptotics or the Plancherel-Rotach type asymptotics do not provide exciting results for these polynomials. Now, our goal is more practical as we will describe below.

The paper [19] achieved two objectives. First, it presents a detailed description of the local asymptotic behavior of these Sobolev-type orthonormal polynomials or discrete Sobolev orthonormal polynomials (dSOP) in terms of a linear combination of special functions — viz. the Bessel functions of the first kind

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{2k+\alpha}. \tag{1.2}$$

The results obtained are valid for measures with bounded and unbounded supports and generalize the works considering specifically chosen measures. Thus, paraphrasing [19], the paper provides "a final and global vision of the Mehler-Heine asymptotics" for these dSOP. Although the obtention of the Mehler-Heine formulae for the dSOP is constructive, many details are omitted. Thus a question arises: what else can we add to this topic?

Obtaining the limit functions in [19, Theorems 1 and 2] requires many calculations. Hence, the answer to the previous question is to provide an algorithm and the corresponding computer program, which would automatically determine the limit functions of these Mehler-Heine formulae — i.e. we used symbolic computation to obtain these limit functions involving Bessel functions. This program is implemented on a computer using the *Mathematica*® language 12.1.1 (also known as Wolfram language).† The program can be downloaded from the `https://w3.ual.es/GruposInv/Tapo/MHAS.nb`.

Numerical algorithms for computation are a necessary and relevant tool when we want to use orthogonal polynomials and special functions in applications. There are nice classic books and articles on this topic — cf. Refs. [1,9,14–16,23]. Besides, there are many others

†The program in the previous versions of *Mathematica*® may not work properly — e.g. we found malfunctions in the 11-th version.