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Superconvergence of Finite Element Approximations of the Two-Dimensional Cubic Nonlinear Schrödinger Equation

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Abstract. The superconvergence of a two-dimensional time-independent nonlinear Schrödinger equation are analyzed with the rectangular Lagrange type finite element of order k. Firstly, the error estimate and superclose property are given in H^1-norm with order $\mathcal{O}(h^{k+1})$ between the finite element solution u_h and the interpolation function u_I by use of the elliptic projection operator. Then, the global superconvergence is obtained by the interpolation post-processing technique. In addition, some numerical examples with the order $k=1$ and $k=2$ are provided to demonstrate the theoretical analysis.

AMS subject classifications: 65N15, 65N30

Key words: Superconvergence, nonlinear Schrödinger equation, finite element method, elliptic projection.

1 Introduction

In this paper, we consider the following two-dimensional time-independent cubic nonlinear Schrödinger equation:

$$\begin{cases} -\Delta u(\mathbf{x})+V(\mathbf{x})u(\mathbf{x})+|u(\mathbf{x})|^2u(\mathbf{x})=f(\mathbf{x}), & \forall \mathbf{x}\in\Omega, \\ u(\mathbf{x})=0, & \forall \mathbf{x}\in\partial\Omega, \end{cases} \tag{1.1}$$

where $i=\sqrt{-1}$ is the complex unit, $\Omega\subset R^2$ be a rectangular domain with smooth boundary $\partial\Omega$, functions $u(\mathbf{x})$ and $f(\mathbf{x})$ are complex-valued, the trapping potential function $V(\mathbf{x})$ is real-valued and non-negative bounded, and suppose a real number $V_0>0$ satisfying

$$V(\mathbf{x})\geq V_0, \quad \forall \mathbf{x}\in\Omega. \tag{1.2}$$

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The Schrödinger equation may describe many physical phenomena in optics, mechanics, plasma physics and so on. In the past several decades, the numerical methods for the Schrödinger equation have been investigated extensively, for example, [1–3] for finite difference methods, [4–14] for finite element methods, [15–18] for local discontinuous Galerkin methods, [19–23] for two-grid methods, and [24–26] for others. In [2], Han et al. proposed a finite-difference scheme for the one-dimensional time-dependent Schrödinger equation. They introduced an artificial boundary condition to reduce the original problem into an initial-boundary value problem in a finite-computational domain, and constructed a finite-difference scheme by the method of reduction of order to solve the reduced problem. In [4], Akrivis et al. presented several implicit Crank-Nicolson Galerkin FEMs for the nonlinear Schrödinger equation. The authors obtained the optimal error estimate with time-step condition $\tau = o(h^{\frac{d}{4}})$, $(d=2,3)$. In [18], Xu and Shu developed a local discontinuous Galerkin method to solve the generalized nonlinear Schrödinger equation and the coupled nonlinear Schrödinger equation, and obtained L^2 stability of the schemes for both of these nonlinear equations. In [20], Jin et al. constructed some two-grid finite element schemes for solving the two-dimensional time-independent nonlinear Schrödinger equation. So then, the solution of the original problem was reduced to the solution of the same problem on a much coarser grid together with the solutions of two linear problems on a fine grid. In [24], Bao et al. studied the performance of time-splitting spectral approximations for general nonlinear Schrödinger equations in the semiclassical regimes, where the Planck constant ε is small.

The superconvergence for the finite element solution has been an active research area in numerical analysis for decades, and many excellent research results have been obtained for elliptic, parabolic, Stokes, Maxwell's and metamaterials problems [27–42]. In [32], Huang et al. considered the time-dependent Maxwell's equations modeling wave propagation in metamaterials, and proved one-order higher global superclose results in the L^2 norm for several semidiscrete and fully discrete schemes using nonuniform cubic and rectangular edge elements. In [40], Wang and Ye obtained a general superconvergent result for finite element approximations of the Stokes problem by using projection methods proposed and analyzed by Wang for the standard Galerkin method. There are also some wonderful superconvergent results for the Schrödinger equation [43–49]. In these literature, the authors considered the superconvergence for the linear or nonlinear time-dependent Schrödinger equation, but the time-independent problems are not involved. In [43], Lin and Liu considered a kind of initial boundary value problem of the Schrödinger equation and presented superconvergent estimates in semi-discrete and fully discrete schemes by the interpolation error theory. In [44], Shi et al. applied the simplest anisotropic linear triangular finite element to solve the nonlinear Schrödinger equation, and provided the superconvergent error estimate in the semi-discrete scheme. In [48], Wang and Chen studied a two-dimensional time-dependent linear Schrödinger equation with the rectangular Lagrange type finite element of order p, and analyzed the superconvergent error estimate in the semi-discrete scheme and the fully discrete scheme respectively.