East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.080621.030921

Vol. **12**, No. 2, pp. 233-246
May 2022

Restrictive Preconditioning for Convection-Diffusion Distributed Control Problems

Wei Feng[1], Zeng-Qi Wang[1,2,3,*], Ruo-Bing Zhong[1] and Galina V. Muratova[4]

[1]*School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China.*
[2]*State Key Laboratory of Functional Materials for Informatics, Shanghai Institute of Microsystem and Information Technology, Chinese Academy of Sciences, 865 Changning Road, Shanghai 200050, China.*
[3]*Ministry of Education Key Lab in Scientific and Engineering Computing, Shanghai Jiao Tong University, Shanghai 200240, China.*
[4]*Laboratory of Computational Mechanics, I.I. Vorovich Institute of Mathematics, Mechanics and Computer Science, Southern Federal University, Rostov-on-Don 344090, Russia.*

Received 8 June 2021; Accepted (in revised version) 3 September 2021.

Abstract. The restrictive preconditioning technique is employed in the preconditioned conjugate gradient and preconditioned Chebyshev iteration methods for the saddle point linear systems arising in convection-diffusion control problems. Utilizing an appropriate approximation of Schur complement, one obtains preconditioned matrix with eigenvalues located in the interval $[1/2, 1]$. The convergence rate of the methods is studied. Unlike the restrictively preconditioned conjugate gradient method, the restrictively preconditioned Chebyshev iteration method is more tolerant to the inexact execution of the preconditioning. This indicates that the preconditioned Chebyshev iteration method is more practical when dealing with large scale linear systems. Theoretical and numerical results demonstrate that the iteration count of the solvers used do not depend the mesh size, the regularization parameter and on the Peclet number.

AMS subject classifications: 65F10, 65N22

Key words: Convection-diffusion distributed control problem, restrictive preconditioning, conjugate gradient method, Chebyshev semi-iteration method.

1. Introduction

We consider the numerical solution of convection-diffusion control problems. The dominating operator in the statement equation is non-selfadjoint and non-symmetric, which

*Corresponding author. *Email addresses:* `williamfeng@alumni.sjtu.edu.cn` (W. Feng), `wangzengqi@sjtu.edu.cn` (Z.-Q. Wang), `icying@sjtu.edu.cn` (R.-B. Zhong), `muratova@sfedu.ru` (G.V. Muratova)

causes difficulties in theoretical analysis and numerical solution of the related optimization problems. This study focuses on the iterative solution of large sparse structured linear systems generated by the discretize-then-optimize procedure. It is worth noting that these systems have symmetric and indefinite saddle point coefficient matrices. In recent years, the numerical solution of the saddle point problems are well developed. The list of the existing iterative methods and preconditioners includes the minimal residual method (MINRES) with a block diagonal preconditioner [20], the generalized minimal residual method (GMRES) with a block triangular preconditioner [14], Uzawa type methods [9,15], Hermitian/Skew-Hermitian splitting type iteration methods [3–6,12], and so on. We refer the reader to [13] for more details.

In order to solve linear system arising in convection-diffusion control problems one usually uses two typical methodologies. Thus Person and Wathen [22] developed block-diagonal and block-triangular preconditioned MINRES and GMRES methods depending on the structure and the properties of saddle point matrices. On the other hand, there are methods specifically constructed for the reduced block two-by-two structure system, such as the non-standard-norm preconditioner [26] and the preconditioned square block preconditioner [1,2]. Axelsson *et al.* [1] showed that the methods mentioned are robust with respect to the discretization mesh size h and the regularization parameter β. However, the number of iterations needed grows when h and β diminish.

In this paper, we apply the restrictively preconditioned conjugate gradient (RPCG) method [7] and the respectively preconditioned Chebyshev iteration (RPCHEB) method [24] to saddle point systems arising in convection-diffusion control problems. It is known that the conjugate gradient method and Chebyshev iteration method are not applicable to the indefinite problems — cf. [23]. However, a restrictive preconditioning makes the methods feasible and effective. Moreover, the convergence rates of restrictively preconditioned methods do not depend on the mesh size, regularization parameter, and even on the Peclet number. In this sense, they are optimal solvers for the linear systems related to the convection-diffusion control problems.

The paper is organized as follows. In Section 2, we recall the restrictive preconditioning technique and the corresponding iteration solvers. Section 3 considers the restrictively preconditioned conjugate gradient method and the restrictively preconditioned Chebyshev iteration method for linear systems achieved by the discrete-then-optimize procedure. The convergence of the methods under consideration is also studied. In addition, we present the convergence rate of the inexact implementation of RPCHEB method. In Section 4, we demonstrate the robustness and efficiency of the new methods. Our conclusions are summarized in Section 5.

2. Restrictively Preconditioned Iteration Methods

The restrictively preconditioning was presented firstly in [7]. It is said that for any non-singular linear system of equations

$$\mathscr{A}\mathbf{x} = \mathbf{b}, \tag{2.1}$$